Def. For Lie algebra L with L-module V say V is *irreducible* when its only submodules are $\{0\}$ and V.

Th: For $L = \mathfrak{sl}(2,F)$, $V = F[X,Y]$, V_m for $0 \le m \in \mathbb{Z}$ as defined before, each V_m is an irreducible L-module.

Th. For $\underline{h} = F\text{-span}\{p = \partial_x, q = x, I\}$, $V = F[X]$ is an irreducible $\underline{h}$-module.

Pf. Exercise.

Th: For the "classical" matrix Lie algebras, $\mathfrak{o}(2\ell, F)$ $\mathfrak{o}(2\ell+1, F)$, $\mathfrak{sp}(2\ell, F)$, $\mathfrak{sl}(n+1, F)$, the representations on column vectors F^m for appropriate m, are each irreducible.

So far we have seen mostly finite dim'l Lie algebras, so here is an ∞-dim'l example.

Def. Let $V = F[t, t^{-1}] = \bigoplus_{m \in \mathbb{Z}} F t^m$ be the vector space of Laurent polynomials in t.
For each $n \in \mathbb{Z}$ define the linear operator $d_n = -t^{n+1} \frac{d}{dt}$ on V and let $L = F\text{-span}\{d_n \mid n \in \mathbb{Z}\}$.
Under commutators this is called the Witt Lie algebra with brackets

$$[d_m, d_n] = (m-n) d_{m+n}.$$

Pf. Compute the action of the commutator on any monomial $t^i \in V$. We get first the formula for

$$d_m(d_n(t^i)) = -t^{m+1}\frac{d}{dt}\left(-t^{n+1}\frac{d}{dt}t^i\right)$$

$$= +t^{m+1}\frac{d}{dt}\left(i\,t^{i+n}\right) = i(i+n)\,t^{i+m+n}$$ so that

$$d_n(d_m(t^i)) = i(i+m)\,t^{i+m+n}$$ and then

$$[d_m,d_n](t^i) = i(n-m)\,t^{i+m+n} = (m-n)\left(-i\,t^{i+m+n}\right)$$

$$= (m-n)\,d_{m+n}(t^i)$$ so

$$[d_m,d_n] = (m-n)\,d_{m+n}. \quad \square$$

<u>Note</u>: Without the minus sign in d_n the factor on the right side would be $(n-m)$.

Also, $d_0 = -t\frac{d}{dt}$ acts diagonally on V by

$$d_0(t^i) = -i\,t^i.$$

A very important Lie algebra in physics is the <u>Virasoro</u> algebra, closely related to the Witt algebra (a 1-dim'l central extension of Witt). Let Vir be the Lie algebra with basis $\{L_m, c \mid m \in \mathbb{Z}\}$ and Lie brackets

$$[c, L_m] = 0 \quad \text{(so } c \text{ is central) and}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{(m^3-m)}{12}\delta_{m,-n}\, c$$

It can be shown that these brackets satisfy the Lie algebra axioms. There is an <u>extensive</u> theory of representations of Vir, deeply related to Conformal Field Theory (CFT), as well as to Vertex Operator Algebras (VOA's) and (affine) Kac-Moody Lie algebras.

Here is a brief peek at an ∞-dim'l Heisenberg Lie algebra. Say char$(\mathbb{F})=0$ to avoid trouble.

Let $H = \mathbb{F}\text{-span}\{h_1, \dots, h_\ell\}$ be abelian, so $[h_i, h_j]=0$. Let

$$\hat{H} = H \otimes_{\mathbb{F}} \mathbb{F}[t,t^{-1}] \oplus \mathbb{F}C \quad \text{with } \mathbb{F}\text{-basis}$$

$$\{h_i \otimes t^m =: h_i(m),\ C \mid 1 \le i \le \ell,\ m \in \mathbb{Z}\} \text{ and}$$

Lie brackets $[C, h_i(m)] = 0$ (so C is central)

$$[h_i(m), h_j(n)] = m\,\delta_{i,j}\,\delta_{m,-n}\,C \quad \text{for } 1 \le i,j \le \ell,\ m,n \in \mathbb{Z}.$$

These brackets can be shown to obey the Lie algebra axioms. The center of $\hat{H}$ is

$$Z(\hat{H}) = \mathbb{F}\text{-span}\{C, h_i(0) \mid 1 \le i \le \ell\}.$$

Let $\hat{H}^{\pm} = \mathbb{F}\text{-span}\{h_i(\pm m) \mid 1 \le i \le l, 0 < m \in \mathbb{Z}\}$.

Then each subspace $\hat{H}^+$ and $\hat{H}^-$ is an abelian Lie subalgebra of $\hat{H}$ because of the $\delta_{m,-n}$ factor on the right side of the main bracket formula.

$[\hat{H}, \hat{H}] = \hat{H}' = \mathbb{F}C$ is 1-dim'l.

$\hat{H} = Z(\hat{H}) \oplus \hat{H}^+ \oplus \hat{H}^-$.

There is a representation of $\hat{H}$ on a vector space of polynomials in infinitely many variables generalizing the rep'n of the 3-dim'l Heisenberg h on $F[X]$. Let $V = S(\hat{H}^-)$ be the "symmetric algebra" of polynomials in the commuting variables

$\{x_{i,m} = h_i(-m) \mid 1 \le i \le l, 0 < m \in \mathbb{Z}\}$.

Th: There is a Lie algebra representation of $\hat{H}$ on $V = S(\hat{H}^-)$, $\phi : \hat{H} \longrightarrow End(V)$, such that

$\phi(h_i(-m))$ is multiplication by $x_{i,m}$ for $0 < m \in \mathbb{Z}$

$\phi(h_i(m))$ is $m\partial_{x_{i,m}}$ is partial diff. w.r.t. $x_{i,m}$ times m

and $\phi(c) = I_V$ is the identity operator.

Pf. Multiplication operators commute with each other, so the restriction of ϕ to $\hat{H}^-$ is a rep'n of $\hat{H}^-$ on V.

The partial diff. operators commute with each other, so the restriction of ϕ to $\hat{H}^+$ is a rep'n of $\hat{H}^+$ on V.

The mult. and partial diff. operators w.r.t. different variables commute so that

for $1 \le i \neq j \le \ell$ or $0 < m, n \in \mathbb{Z}$ with $m \neq n$,
$[h_i(m), h_j(-n)] = m\,\delta_{ij}\,\delta_{m,n}\,C = 0$ and
$[m\,\partial_{x_{i,m}}, x_{j,n}] = 0$ as required. It remains to check the case when $i=j$ and $m=n$, where
$[h_i(m), h_i(-m)] = m\,\delta_{i,i}\,\delta_{m,m}\,C = m\,C$.
We must show that on V the commutator
$[m\,\partial_{x_{i,m}}, x_{i,m}] = m\,I_V$. But these operators only affect one variable, $x = x_{i,m}$, and we already know from $\mathfrak{h}$ acting on $F[x]$ that $[\partial_x, x] = I$ so this bracket relation is true on V. The identity operator is central and represents C.

We have not said what operators should represent $h_i(0)$, $1 \le i \le \ell$, but since these are central in $\hat{H}$, they can be represented by any scalar operators, say $\phi(h_i(0)) = \mu_i I_V$ for any choices of $\mu_i \in \mathbb{F}$. $\square$

The rep'n $V = S(\hat{H}^-)$ is called a <u>Fock-space</u> in honor of the math. physicist, Vladimir A. Fock (1898-1974), last name is spelled "ФОК" in Russian. (See Wikipedia entry.) The constant polynomial $1 \in V$ is called the <u>vacuum</u> vector, $h_i(-m)$, $0<m$, are called <u>creation operators</u>, while $h_i(m)$, $0<m$, are <u>annihilation operators</u>. The actions of these operators <u>create states</u> from the vacuum or <u>annihilate states</u>, respectively.

The simplest case when $\ell=1$ can be seen to give a <u>combinatorial</u> description of a basis for $\mathbb{Z}^{+}$-graded space $S(\hat{H}^{-}) = \bigoplus_{m=0}^{\infty} V_m$ where V_m is spanned by monomials

$$\left\{ h_1(-m_1)\, h_1(-m_2) \cdots h_1(-m_r)\, 1 \;\middle|\; \begin{array}{l} m_1 \geq m_2 \geq \cdots \geq m_r > 0, \\ m = m_1 + m_2 + \cdots + m_r \end{array} \right\}$$

of total degree m.

This shows that $\dim(V_m) = p(m)$ the classical partition function.