We can give a basis of $\mathfrak{g}$ as follows:

$:a_i a_j: = a_i a_j = -a_j a_i$ for $1 \le i < j \le m$, $\quad m(m-1)/2$

$:a_i^* a_j^*: = a_i^* a_j^* = -a_j^* a_i^*$ for $1 \le i < j \le m$, $\quad m(m-1)/2$

$:a_i a_j^*: = a_i a_j^* - \frac{1}{2}\delta_{ij} = -a_j^* a_i + \frac{1}{2}\delta_{ij}$, $1 \le i, j \le m$, $\quad m^2$

$\dim(\mathfrak{g}) = 2m^2 - m$

Let's calculate

$$[:ab:, :cd:] = [ab, cd] = abcd - cdab$$

$$\begin{aligned} &= abcd + acbd \\ &\quad - acbd - cabd \\ &\quad + cabd + cadb \\ &\quad - cadb - cdab \end{aligned}$$

$$= a(bc+cb)d - (ac+ca)bd + ca(bd+db) - c(ad+da)b$$

$$= (b,c)ad - (a,c)bd + (b,d)ca - (a,d)cb$$

$$= (b,c)\left(:ad: + \tfrac{1}{2}(a,d)\right) - (a,c)\left(:bd: + \tfrac{1}{2}(b,d)\right) + (b,d)\left(:ca: + \tfrac{1}{2}(c,a)\right) - (a,d)\left(:cb: + \tfrac{1}{2}(c,b)\right)$$

$= (b,c):ad: - (a,c):bd: - (b,d):ac: + (a,d):bc:$ $\boxed{275}$

$+ \frac{1}{2}\Big((b,c)(a,d) - (a,c)(b,d) + (b,d)(c,a) - (a,d)(c,b) \Big)$ So

Lemma: $\forall :ab:, :cd: \in g$, we have

$[:ab:, :cd:] = (b,c):ad: - (a,c):bd: - (b,d):ac: + (a,d):bc:$

$\in g$ so g is a Lie algebra under commut.

Cor: Letting $h_i = :a_i a_i^*:$ for $1 \le i \le m$, we have $[h_i, h_j] = 0$.

Pf. $[h_i, h_j] = [:a_i a_i^*:, :a_j a_j^*:] = (a_i^*, a_j):a_i a_j^*: -$

$(a_i, a_j):a_i^* a_j^*: - (a_i^*, a_j^*):a_i a_j: + (a_i, a_j^*):a_i^* a_j:$

$= \delta_{ij} :a_i a_j^*: + \delta_{ij} :a_i^* a_j: = \delta_{ij} :a_i a_i^*: + \delta_{ij} :a_i^* a_i: = 0.$

$\square$

Def. Let $H = \mathrm{span}\{h_i = :a_i a_i^*: \in g \mid 1 \le i \le m\}$.

Compute the action of ad_{h_i} on basis vectors of $\mathfrak{g}$. Show they act simultaneously diagonally with e-values given in terms of linear functionals $\varepsilon_i \in H^*$ where $\varepsilon_i(h_j) = \delta_{ij}$.

$$[h_i, :a_j a_k:] = [:a_i a_i^*:, :a_j a_k:] = (a_i^*, a_j) :a_i a_k: -$$

$$(a_i, a_j) :a_i^* a_k: - (a_i^*, a_k) :a_i a_j: + (a_i, a_k) :a_i^* a_j:$$

$$= \delta_{ij} :a_i a_k: - \delta_{ik} :a_i a_j:$$

$$= \delta_{ij} :a_j a_k: - \delta_{ik} :a_k a_j: = (\delta_{ij} + \delta_{ik}) :a_j a_k:$$

$$= (\varepsilon_j + \varepsilon_k)(h_i) :a_j a_k: \qquad \text{for } 1 \le j < k \le m.$$

Similar calculations (exercise!) show:

$[h_i, :a_j^* a_k^*:] = -(\varepsilon_j + \varepsilon_k)(h_i) :a_j^* a_k^*:$ and

$[h_i, :a_j a_k^*:] = (\varepsilon_j - \varepsilon_k)(h_i) :a_j a_k^*: \quad (1 \le j, k \le m)$

This gives a Cartan decomposition

$$g = H \oplus \bigoplus_{\alpha \in \Phi} g_\alpha \quad \text{for } \Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le m\}$$

where $g_\alpha = \operatorname{span}\{X_\alpha\}$ and

$X_\alpha = :a_i a_j: \text{ if } \alpha = \varepsilon_i + \varepsilon_j \quad (1 \le i < j \le m)$

$X_\alpha = :a_i^* a_j^*: \text{ if } \alpha = -\varepsilon_i - \varepsilon_j \quad (1 \le i < j \le m)$

$X_\alpha = :a_i a_j^*: \text{ if } \alpha = \varepsilon_i - \varepsilon_j \quad (1 \le i \ne j \le m).$

<u>Th.</u> $g \cong o(2m,F)$ is the simple Lie alg.
of type D_m.

<u>Pf.</u> The root system Φ of g matches the root system of $o(2m,F)$, type D_m, and the explicit commutators of root vectors x_α and Cartan subalgebra basis vectors, $h_i, 1 \le i \le m$, match the commutators of matrices in the "std." basis of $o(2m,F)$. To see this more clearly, go back to Humphreys, page 3, and the definition of $o(2m,F) = \left\{ X = \begin{bmatrix} m & n \\ p & q \end{bmatrix} \in F_{2m}^{2m} \middle| JX = -X^{Tr} J \right\}$ for $J = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$ where $m,n,p,q \in F_m^m$ must satisfy

$$\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}\begin{bmatrix} m & n \\ p & q \end{bmatrix} = -\begin{bmatrix} m^{Tr} & p^{Tr} \\ n^{Tr} & q^{Tr} \end{bmatrix}\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$$ so

$$\begin{bmatrix} p & q \\ m & n \end{bmatrix} = \begin{bmatrix} -p^{Tr} & -m^{Tr} \\ -q^{Tr} & -n^{Tr} \end{bmatrix}$$ says

$p^{Tr} = -p$, $n^{Tr} = -n$ are anti-symmetric, and $q = -m^{Tr}$ (same as $m = -q^{Tr}$).

Thus, $o(2m, F) = \left\{ \left[\begin{array}{c|c} m & n \\ \hline p & -m^{Tr} \end{array}\right] \in F_{2m}^{2m} \,\middle|\, n^{Tr} = -n,\ p^{Tr} = -p \right\}$

has basis $\{E_{ii} - E_{m+i,m+i} = d_i \mid 1 \le i \le m\}$ (diagonal)

$\cup \{E_{ij} - E_{m+j,m+i} \mid 1 \le i \ne j \le m\} \cup \{E_{i,m+j} - E_{j,m+i} \mid 1 \le i < j \le m\}$

$\cup \{E_{m+i,j} - E_{m+j,i} \mid 1 \le i < j \le m\}$.

Check (if you dare) that the isomorphism between $\mathfrak{g}$ and $\mathfrak{o}(2m, F)$ comes from the correspondence of basis vectors:

$h_i \leftrightarrow d_i$, $\quad :a_i a_j: \leftrightarrow E_{i,m+j} - E_{j,m+i}$,

$:a_i^* a_j^*: \leftrightarrow E_{m+i,j} - E_{m+j,i}$ and

$:a_i a_j^*: \leftrightarrow E_{ij} - E_{m+j,m+i}$ $\quad (1 \le i \ne j \le m)$. □

<u>Th</u>: CM_m is a $\mathfrak{g}$-module where the action of $\mathfrak{g}$ on CM_m is by left multiplication since $\mathfrak{g} \le Cliff_m$ and CM_m is a left $Cliff_m$ module

<u>Goals</u>: ① Decompose CM_m into irred. $\mathfrak{g}$-modules,

② For each $\mathfrak{g}$-module (irred.) in CM_m, understand its "weight space" decomposition w.r.t. H action (which is diagonal).

③ Understand the action of $\mathfrak{g}$ on $V \cong F^{2m}$ as the "natural" representation.

Do ③ first: what is, $\forall a, b, c \in V$, where

$$(:ab:)c \stackrel{\text{def}}{=} [:ab:, c] = [ab, c] = abc - cab$$

$$= abc + acb$$
$$- acb - cab$$

$$= a(bc+cb) - (ac+ca)b$$

$$= (b,c)a - (a,c)b \in V$$

gives a rep'n of $\mathfrak{g}$ on V (exercise).