Since commutators in any associative alg.
obey the Jacobi identity, we have

$$[[:ab:, :cd:], e] = [[:ab:, e], :cd:] + [:ab:, [:cd:, e]]$$

$$= [(b,e)a - (a,e)b, :cd:] + [:ab:, (d,e)c - (c,e)d]$$

$$= (b,e)[a, :cd:] - (a,e)[b, :cd:]$$
$$+ (d,e)[:ab:, c] - (c,e)[:ab:, d]$$

$$= -(b,e)[:cd:, a] + (a,e)[:cd:, b] + (d,e)[:ab:, c] - (c,e)[:ab:, d]$$

$$= -(b,e)((d,a)c - (c,a)d) + (a,e)((d,b)c - (c,b)d)$$
$$+ (d,e)((b,c)a - (a,c)b) - (c,e)((b,d)a - (a,d)b)$$

$$= ((b,c)(d,e) - (b,d)(c,e))a + ((a,d)(c,e) - (a,c)(d,e))b +$$

$$((a,e)(b,d)-(a,d)(b,e))c+((a,c)(b,e)-(a,e)(b,c))d$$

which should equal (by Lemma on page 275)

$$[(b,c){:}ad{:}-(a,c){:}bd{:}-(b,d){:}ac{:}+(a,d){:}bc{:}\,,e].$$

Perhaps all we needed to say was that

$$[[{:}ab{:},{:}cd{:}],e]=[[{:}ab{:},e],{:}cd{:}]+[{:}ab{:},[{:}cd{:},e]]$$

$$=[{:}ab{:},[{:}cd{:},e]]-[{:}cd{:},[{:}ab{:},e]] \quad \text{so that}$$

$$[{:}ab{:},{:}cd{:}]\cdot e=({:}ab{:})\cdot(({:}cd{:})\cdot e)-({:}cd{:})\cdot(({:}ab{:})\cdot e)$$

makes V a $\mathfrak{g}$-module under this action.

What are the weights of this $\mathfrak{g}$-module?

Compute $h_i = {:}a_i a_i^*{:}$ action on a_j and on a_j^*.

$$(:a_i a_i^*:)\cdot a_j = [:a_i a_i^*:, a_j] =$$

$$(a_i^*, a_j)a_i - (a_i, a_j)a_i^* = \delta_{ij} a_i = \delta_{ij} a_j = \varepsilon_j(h_i)a_j$$

so a_j has weight $\varepsilon_j \in H^*$.

$$(:a_i a_i^*:)\cdot a_j^* = [:a_i a_i^*:, a_j^*] = (a_i^*, a_j^*)a_i - (a_i, a_j^*)a_i^*$$

$$= -\delta_{ij} a_i^* = -\delta_{ij} a_j^* = -\varepsilon_j(h_i)a_j^*$$ so

a_j^* has weight $-\varepsilon_j \in H^*$.

The set of all weights of the g-module V is $\{\pm\varepsilon_j \mid 1 \leq j \leq m\}$.

Compare this with $o(2m,F)$ action on F^{2m}.

In $o(2m,F)$ the Cartan subalgebra of diagonal matrices has basis $\{d_i = E_{ii} - E_{m+i,m+i}\}$

In F^{2m} define the natural (std.) basis $S = \{e_j \in F^{2m} \mid 1 \le j \le 2m\}$. Then $\forall\, 1 \le j \le 2m$,

$$d_i \cdot e_j = (E_{ii} - E_{m+i,m+i})\, e_j = \mathrm{Col}_j(E_{ii} - E_{m+i,m+i})$$

$$= \delta_{ij}\begin{cases} e_j & \text{if } 1 \le j \le m \\ -e_{j+m} & \text{if } 1 \le j \le m \end{cases} = \begin{cases} \varepsilon_j(d_i)\, e_j & \text{if } 1 \le j \le m \\ -\varepsilon_j(d_i)\, e_{j+m} & \text{if } 1 \le j \le m \end{cases}$$

where $\varepsilon_j(d_i) = \delta_{ij}$ defines $\varepsilon_j \in H^*$. So set of weights is $\{\pm\varepsilon_j \mid 1 \le j \le m\}$.

Now let's look at $\mathfrak{g}$-module action on $\underline{286}$
CM_m. Let $\mathbb{1} = 1 + \mathfrak{L}_m$ = "vacuum vector" = vac.

$CM_m = \operatorname{span}\{(a_1^*)^{k_1} \cdots (a_m^*)^{k_m} \cdot \mathbb{1} \mid k_i = 0, 1\}$

Compute action on these basis vectors of

$h_i = :a_i a_i^*: = -a_i^* a_i + \frac{1}{2}\delta_{ii} = -a_i^* a_i + \frac{1}{2}$.

$h_i \cdot \mathbb{1} = (-a_i^* a_i + \frac{1}{2}) \cdot \mathbb{1} = \frac{1}{2} \cdot \mathbb{1}$ for all $1 \leq i \leq m$

$= \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_m)(h_i)\, \mathbb{1}$ so

$\mathbb{1}$ has weight $\lambda = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_m)$.

Compute $h_i \cdot (a_j^* \cdot \mathbb{1}) = [h_i, a_j^*]\mathbb{1} + a_j^* \cdot (h_i \cdot \mathbb{1})$

$= \varepsilon_j(h_i)\, a_j^* \cdot \mathbb{1} + a_j^* \cdot \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_m)(h_i) \cdot \mathbb{1}$

$= \left(\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_m) - \varepsilon_j\right)(h_i)\, a_j^* \cdot 1$

$= \frac{1}{2}(\varepsilon_1 + \cdots - \varepsilon_j + \cdots + \varepsilon_m)(h_i)\, a_j^* \cdot 1$ says

$a_j^* \cdot 1$ has weight $\lambda - \varepsilon_j$.

In general $(a_1^*)^{k_1} \cdots (a_m^*)^{k_m} \cdot 1$ has weight $\lambda - \sum_{j=1}^{m} k_j \varepsilon_j$ so the set of all weights in $C\mathcal{M}_m$ is $\left\{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \cdots \pm \varepsilon_m)\right\}$ with indep. choices of all $\pm$ signs, $2^m = \dim(C\mathcal{M}_m)$