$= \left(\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_m) - \varepsilon_j\right)(h_i)\, a_j^* \cdot 1$

$= \frac{1}{2}(\varepsilon_1 + \cdots - \varepsilon_j + \cdots + \varepsilon_m)(h_i)\, a_j^* \cdot 1$ says

$a_j^* \cdot 1$ has weight $\lambda - \varepsilon_j$.

In general $(a_1^*)^{k_1} \cdots (a_m^*)^{k_m} \cdot 1$ has weight

$\lambda - \sum_{j=1}^{m} k_j \varepsilon_j$ so the set of all weights

in CM_m is $\left\{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \cdots \pm \varepsilon_m)\right\}$ with

indep. choices of all $\pm$ signs, $2^m = \dim(CM_m)$

__Def.__ For $i = 0, 1$, let $CM_m^i =$

$\mathrm{span}\left\{(a_1^*)^{k_1} \cdots (a_m^*)^{k_m} 1 \,\middle|\, \sum_{j=1}^{m} k_j \equiv i \pmod 2\right\}$.

<u>Th</u>. $CM_m = CM_m^0 \oplus CM_m^1$ is a direct sum decomposition of CM_m as irreducible g-modules.

<u>Pf</u>. This decomposition by the parity of the number of creation operators applied to the vacuum vector 1, is preserved by the action of the quadratic operators $:ab: \in g$. For $i \neq j$, $:a_i^* a_j^*: \cdot 1 = a_i^* a_j^* 1 \in CM_m^0$ and for $k \neq l$,

$$:a_k^* a_l^*: \cdot (a_i^* a_j^* 1) = a_k^* a_l^* a_i^* a_j^* 1 =$$

$$\pm(\delta_{ki}-1)(\delta_{kj}-1)(\delta_{li}-1)(\delta_{lj}-1)\, a_k^* a_l^* a_i^* a_j^* 1$$

since a factor of $(a_i^*)^2 = 0$ or $(a_j^*)^2 = 0$ occurs after anti-commuting if $k \in \{i,j\}$ or $l \in \{i,j\}$.

Also, the anti-commutation relations in Cliffm show that the action of $:a_i a_j^*:$ for $i \neq j$ on a general basis vector of CM_m gives 0 if $k_j = 1$ or if $k_i = 0$, but when $k_i = 1$ and $k_j = 0$ it has the effect of removing a_i^* and inserting a_j^*, preserving the parity, and at most multiplying the vector by ± 1.

<u>EX.</u> $:a_1 a_2^*: \cdot (a_1^* a_3^* 1) = a_1 a_2^* a_1^* a_3^* 1 =$

$-a_2^* a_1 a_1^* a_3^* 1 = -a_2^* (-a_1^* a_1 + 1) a_3^* 1$

$= a_2^* a_1^* a_1 a_3^* 1 - a_2^* a_3^* 1 =$

$= a_2^* a_1^* (-a_3^* \underbrace{a_1) 1}_{=0} - a_2^* a_3^* 1 = -a_2^* a_3^* 1.$

Similarly, the action of $:a_i a_j:$ $\in \mathfrak{g}$ on a general vector gives 0 if $k_i = 0$ or $k_j = 0$ since the annihilation operators a_i or a_j will anti-commute passed the creation operators with different subscripts and kill 1.

If $k_i = 1 = k_j$ then the action removes both a_i^* and a_j^*, leaving ± 1 times a basis vector with $k_i = 0 = k_j$. This preserves parity.

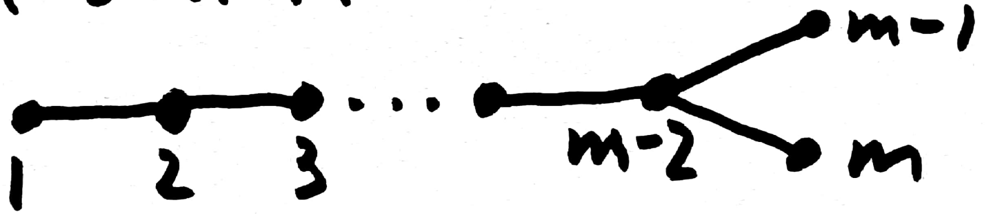
With the partial order on weights coming from the choice of simple roots

$$\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{m-1} = \varepsilon_{m-1} - \varepsilon_m, \alpha_m = \varepsilon_{m-1} + \varepsilon_m\}$$

we find $\lambda = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_m)$ is the highest weight in CM_m^0 (of the vacuum vector, 1) and

$\lambda - \varepsilon_m$ is the highest weight in CM_m^1 (of the vector, $a_m^* \mathbb{1}$).

Since $\langle \lambda, \alpha_j \rangle = \delta_{jm}$ and $\langle \lambda - \varepsilon_m, \alpha_j \rangle = \delta_{j(m-1)}$ we have $\lambda = \lambda_m$ and $\lambda - \varepsilon_m = \lambda_{m-1}$ are the fundamental weights corresponding to the vertices m and $m-1$ in the D_m Dynkin diagram:



. □

__Note__: The natural $\mathfrak{g}$-module $U \cong F^{2m}$ with weights $\{\pm \varepsilon_j \mid 1 \leq j \leq m\}$ has highest weight ε_1 (for the vector a_1) and $\lambda_1 = \varepsilon_1$ since $\langle \varepsilon_1, \alpha_j \rangle = \delta_{j1}$.

Weyl algebras: Char(F) ≠ 2

Let V be a vector space over F with an anti-symm. bil. form $(\cdot,\cdot): V \times V \to F$.
Let $\mathrm{Weyl}(V, (\cdot,\cdot))$ be the assoc. alg. with 1 generated by V with relations

$$v_1 v_2 - v_2 v_1 = (v_1, v_2)1 \quad \forall v_1, v_2 \in V.$$

For $(\cdot,\cdot)$ to be non-degenerate we must have $\dim(V) = 2m$ is even. Assume V has a basis $B = \{a_1, \dots, a_m, a_1^*, \dots, a_m^*\}$ s.t. the matrix of the form w.r.t. B is $M = \left[\begin{array}{c|c} 0 & I_m \\ \hline -I_m & 0 \end{array}\right] = J$ (Humphreys).

So the relations are:

$a_i a_j - a_j a_i = 0 = a_i^* a_j^* - a_j^* a_i^*$ for $1 \le i, j \le m$
and

$a_i a_j^* - a_j^* a_i = \delta_{ij}$ for $1 \le i, j \le m$ so $a_j^* a_i - a_i a_j^* = -\delta_{ij}$ is implied. Note that the case of $i = j$ in the first relations gives no information about a_i^2 and $(a_i^*)^2$.

<u>Th.</u> $\mathrm{Weyl}_m = \mathrm{span}\{(a_1^*)^{k_1} \cdots (a_m^*)^{k_m} a_1^{l_1} \cdots a_m^{l_m} I \mid 0 \le k_i, l_i\}$ is ∞-dim'l.

<u>Def.</u> Let $\mathcal{L}_m$ = left ideal of Weyl_m gen. by $a_1, \ldots, a_m$ so the basis vectors with any $l_i > 0$ are in $\mathcal{L}_m$.

<u>Def.</u> Let $WM_m = \mathrm{Weyl}_m / \mathcal{L}_m$ which has a basis whose coset representatives are $\{(a_1^*)^{k_1} \cdots (a_m^*)^{k_m} I \mid 0 \le k_i \in \mathbb{Z}\}$ and has vacuum vector $\mathbb{1} = I + \mathcal{L}_m$.

Th. WM_m is a $Weyl_m$-module with left mult. as the action on WM_m. The a_i act as annihilation operators, a_i^* act as creation ops.

Note. These are Bosonic operators since $a_i^n \neq 0$, $\forall n \geq 1$, $(a_i^*)^n \neq 0$, $\forall n \geq 1$ and generators have commutation relations.

Let $V = \text{span}\{a_i, a_i^* \mid 1 \leq i \leq m\}$ as before, but $\forall a, b \in V$ let $:ab: = \frac{1}{2}(ab + ba) =: ba:$ be the "bosonic" normally ordered product. So $ab - ba = (a,b)1$ says $:ab: = \frac{1}{2}(ab + ab - (a,b)1)$ $= ab - \frac{1}{2}(a,b)1$ $= \frac{1}{2}(ba + (a,b)1 + ba) =$ $= ba + \frac{1}{2}(a,b)1$. Now $:a_i a_i: \neq 0 \neq :a_i^* a_i^*:$

Def. Let $\mathfrak{g} = \text{span}\{:ab: = \frac{1}{2}(ab+ba) \mid a, b \in U\}$.

Then $\mathfrak{g}$ is a subspace of Weyl_m with basis $\{:a_i a_j: \mid 1 \le i \le j \le m\} \cup \{:a_i^* a_j^*: \mid 1 \le i \le j \le m\}$ $\cup \{:a_i a_j^*: \mid 1 \le i, j \le m\}$ so, counting basis vectors, get $\dim(\mathfrak{g}) = \frac{(m)(m+1)}{2} + \frac{m(m+1)}{2} + m^2 = 2m^2 + m$ $= \dim(sp(2m, F))$.

Lemma. (Exercise) Find $[:ab:, :cd:]$ formula as a lin. comb. from $\mathfrak{g}$ to get $\mathfrak{g}$ is a Lie alg.

Def. Let $h_i = -:a_i a_i^*:$ for $1 \le i \le m$, $H = \text{span}\{h_i\}$, $H^* = \text{span}\{\varepsilon_i\}$ s.t. $\varepsilon_i(h_j) = \delta_{ij}$.

Cor. $\forall\ 1 \le i, j \le m$, $[h_i, h_j] = 0$.

<u>Cor</u>: $[h_i, :a_j a_k:] = (\varepsilon_j + \varepsilon_k)(h_i) :a_j a_k:$, $j \le k$,

$[h_i, :a_j^* a_k^*:] = -(\varepsilon_j + \varepsilon_k)(h_i) :a_j^* a_k^*:$, $j \le k$, and

$[h_i, :a_j a_k^*:] = (\varepsilon_j - \varepsilon_k)(h_i) :a_j a_k^*:$ for $1 \le j,k \le m$,

so $g = H \oplus \bigoplus_{\alpha \in \Phi} g_\alpha$ where $\Phi = \{\pm\varepsilon_j \pm \varepsilon_k \mid 1 \le j \le k \le m\}$

but $0 \notin \Phi$, and

$g_\alpha = F :a_j a_k:$ for $\alpha = \varepsilon_j + \varepsilon_k$ with $1 \le j \le k \le m$,

$g_\alpha = F :a_j^* a_k^*:$ for $\alpha = -\varepsilon_j - \varepsilon_k$ with $1 \le j \le k \le m$,

$g_\alpha = F :a_j a_k^*:$ for $\alpha = \varepsilon_j - \varepsilon_k$ with $1 \le j \ne k \le m$.

<u>Th</u>: $g \cong sp(2m, F)$ where the isomorphism can be made to match corresponding root vectors in g with specific matrices in the $sp(2m, F)$ basis.

<u>Note</u>: The representation of $\mathfrak{g}$ on WM_m is ∞-dim'l, but $1 \in WM_m$ is a highest weight vector w.r.t. simple roots (type C_m)

$\Delta = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \ldots, \alpha_{m-1} = \varepsilon_{m-1} - \varepsilon_m, \alpha_m = 2\varepsilon_m\}$

and we can compute the weight λ as follows.

$h_i \cdot 1 = -:a_i a_i^*: 1 = -(a_i^* a_i + \frac{1}{2}) 1 = -\frac{1}{2} \cdot 1$ for $1 \le i \le m$

would say $\lambda = -\frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_m)$ so $\lambda(h_i) = -\frac{1}{2}$ for $1 \le i \le m$.

This might need modification if our def. of h_i is off by a scalar! <u>DONE</u>.

<u>Th</u>: $WM_m = WM_m^0 \oplus WM_m^1$ is a decomp. into irred. $\mathfrak{g}$-modules, where WM_m^i for $i = 0, 1$ is a parity decomposition as before.

We may also obtain the natural representation of $\mathfrak{g}$ on $V \cong F^{2m}$ as follows.

Define left action of $\mathfrak{g}$ on V by

$$
\begin{aligned}
(:ab:)\cdot c &= [:ab:, c] = [ab, c] = abc - cab \\
&= abc - acb \\
&\qquad + acb - cab \\
&= a(bc - cb) + (ac - ca)b \\
&= (b,c)a + (a,c)b \in V.
\end{aligned}
$$

Check this gives a Lie alg. rep'n of $\mathfrak{g}$ on V as on page 283.

__Exercise__: Find $h_i \cdot a_j$ and $h_i \cdot a_j^*$ to get weights of these vectors. If we want $a_j \longleftrightarrow \varepsilon_j$ and $a_j^* \longleftrightarrow -\varepsilon_j$ we need $h_i = -:a_i a_i^*:$