A brief introduction to Kac-Moody Lie Alg's.

In Humphreys' book, Section 18, he gives

Serre's Th. Let L be a fin. dim'l semisimple Lie algebra over $\mathbb{C}$ with Cartan subalgebra H, root system Φ and base $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Let $A = [a_{ij}]$ be the $\ell \times \ell$ Cartan matrix where
$$a_{ij} = \langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \alpha_i(h_j).$$
Let $e_i \in L_{\alpha_i}$, $f_i \in L_{-\alpha_i}$ and $h_i = [e_i, f_i] \in H$ be chosen so that they form a "std." basis of a subalg. isomorphic to $sl(2, \mathbb{C})$ in L. Then L is generated as a Lie algebra by $\{e_i, f_i, h_i \mid 1 \le i \le \ell\}$ with the following relations:

(S1) $[h_i, h_j] = 0$, $1 \le i, j \le \ell$,

(S2) $[e_i, f_j] = \delta_{ij} h_i$, $1 \le i, j \le \ell$,

(S3) $[h_i, e_j] = \alpha_j(h_i) e_j$, $[h_i, f_j] = -\alpha_j(h_i) f_j$, "

(S^+_{ij}) $(\mathrm{ad}_{e_i})^{1-a_{ji}} e_j = 0$ for $i \neq j$,

(S^-_{ij}) $(\mathrm{ad}_{f_i})^{1-a_{ji}} f_j = 0$ for $i \neq j$.

This says that the Cartan matrix A gives L by generators and relations. We classified those Cartan matrices s.t. L is fin. dim'l semisimple, so those A are called "finite type." What if more general Cartan matrices are used?

In 1968, Victor Kac (in Russia) and Robert Moody (in Canada) independently defined and started to study the infinite dim'l "Kac-Moody" Lie algebras coming from more general types of Cartan matrix $A=[a_{ij}]$.

Def. A generalized Cartan matrix (GCM) is an integral matrix whose entries satisfy:

① $a_{ii}=2$, ② $0 \geq a_{ij} \in \mathbb{Z}$ for $i \neq j$,

③ $a_{ij}=0$ iff $a_{ji}=0$ for $i \neq j$.

Def: The Lie algebra $L=L(A)$ associated to GCM A is the Lie alg. gen. by $\{e_i, f_i, h_i \mid 1 \leq i \leq l\}$ subject to the relations $(S1)$, $(S2)$, $(S3)$, (S^+_{ij}), (S^-_{ij}).

Ex. The simplest GCM not of fin. type is $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$ (affine type). The "affine" KM algebra $L(A)$ has a nice description in terms of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ as follows.

Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ where $[c, \tilde{\mathfrak{g}}] = 0$ so c is central, and $\forall n \in \mathbb{Z}$, let $x \otimes t^n = x(n)$ for $x \in \mathfrak{g}$ have Lie brackets

$$[x(m), y(n)] = [x,y](m+n) + m\langle x,y\rangle \delta_{m,-n}\, c$$

where $\langle x,y\rangle$ is the Killing form on $\mathfrak{g}$ normalized so that it induces a bil. form on H^* and $\langle \theta, \theta\rangle = 2$ for highest root θ of $\mathfrak{g}$.

For $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ with std basis $\{e, f, h\}$ and scaled Killing form $\langle x,y\rangle = Tr(xy)$ for 2×2 trace zero matrices $x, y \in \mathfrak{g}$, we have $\langle e,f\rangle = 1$, $\langle h,h\rangle = 2$ and others are 0.

So, for example, $\forall m, n \in \mathbb{Z}$,

$$[e(m), f(n)] = [e,f](m+n) + m\langle e,f\rangle \delta_{m,-n} c$$
$$= h(m+n) + m\,\delta_{m,-n} c$$

and

$$[h(m), h(n)] = [h,h](m+n) + m\langle h,h\rangle \delta_{m,-n} c$$
$$= 2m\,\delta_{m,-n} c$$

We see that $\{h(m), c \mid 0 \neq m \in \mathbb{Z}\}$ spans a Heisenberg Lie subalg. of $\tilde{\mathfrak{g}}$.

It is not hard to see the Cartan subalg. of $\tilde{\mathfrak{g}}$ has basis $\{h(0), c\}$ but ad_c is the zero op. and $[h(0), x(n)] = [h,x](n)$ so $\mathrm{ad}_{h(0)}$ is diag.able

$[h(0), e(n)] = [h,e](n) = 2e(n)$

$[h(0), h(n)] = 0$

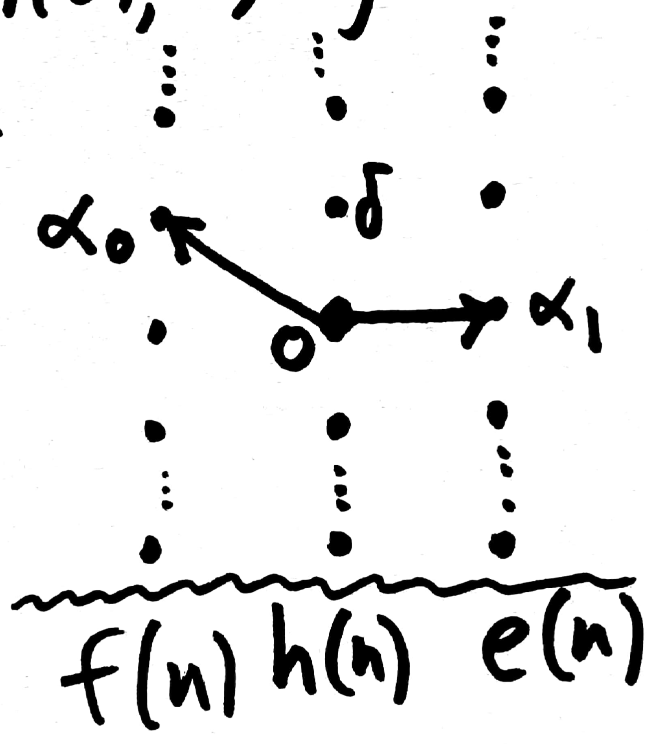
$[h(0), f(n)] = [h,f](n) = -2f(n)$

so $\mathrm{ad}_{h(0)}$ only distinguishes the three e.values of ad_h on $\mathfrak{g}$. Because $\det(A)=0$, we have to add another operator which detects the "mode number" n.

<u>Def</u>. Let $d = -t\frac{d}{dt}$ which acts on the ring of Laurent polys. $\mathbb{C}[t,t^{-1}]$ by $d \cdot t^n = -nt^n$.

Def. Let $\hat{g} = \tilde{g} \oplus \mathbb{C}d$ with additional brackets $[c,d]=0$, $[d, X(n)] = -n\,X(n)$ $= -t\frac{d}{dt}(X \otimes t^n)$.

Th. $\hat{g}$ is a Lie alg. with Cartan subalg. $H = \mathrm{span}\{h(0), c, d\}$ and the root system Φ w.r.t. H is:



$\{m\delta \mid 0 \neq m \in \mathbb{Z}\} \cup \{n\delta \pm \alpha_1 \mid n \in \mathbb{Z}\}$,

$$\hat{g} = H \oplus \bigoplus_{\alpha \in \Phi} \hat{g}_\alpha,$$

$$g_\alpha = \begin{cases} \mathbb{C}e(n) & \text{if } \alpha = n\delta + \alpha_1 \\ \mathbb{C}h(m) & \text{if } \alpha = m\delta \\ \mathbb{C}f(n) & \text{if } \alpha = n\delta - \alpha_1 \end{cases}$$

where
$\delta = \alpha_0 + \alpha_1$,
$\Delta = \{\alpha_0, \alpha_1\}$

Note. Since $\delta = \alpha_0 + \alpha_1$,

$$\Phi = \{m\alpha_0 + m\alpha_1 \mid 0 \neq m \in \mathbb{Z}\} \cup \{n\alpha_0 + (n \pm 1)\alpha_1 \mid n \in \mathbb{Z}\} = \Phi^+ \cup \Phi^-$$

splits into positive or negative roots according to whether the coefficients of $\alpha = k_0\alpha_0 + k_1\alpha_1$ have $k_0, k_1 \geq 0$ or $k_0, k_1 \leq 0$.

The α_1-string through α_0 is $\{\alpha_0, \alpha_0 + \alpha_1, \alpha_0 + 2\alpha_1\}$ so $\langle \alpha_0, \alpha_1 \rangle = r - q = 0 - 2 = -2$ and the α_0-string through α_1 is $\{\alpha_1, \alpha_1 + \alpha_0, \alpha_1 + 2\alpha_0\}$ so $\langle \alpha_1, \alpha_0 \rangle = -2$. The Cartan matrix is

$$A = [\langle \alpha_i, \alpha_j \rangle] = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

The Weyl group $W = \langle r_0, r_1 \mid r_0^2 = 1 = r_1^2 \rangle$ is the infinite dihedral group, D_∞, since

$r_0(\alpha_0) = -\alpha_0$, $r_0(\alpha_1) = \alpha_1 - \langle \alpha_1, \alpha_0 \rangle \alpha_0 = \alpha_1 + 2\alpha_0$

$r_1(\alpha_0) = \alpha_0 - \langle \alpha_0, \alpha_1 \rangle \alpha_1 = \alpha_0 + 2\alpha_1$, $r_1(\alpha_1) = -\alpha_1$

Then $r_0(\delta) = r_0(\alpha_0 + \alpha_1) = -\alpha_0 + \alpha_1 + 2\alpha_0 = \alpha_0 + \alpha_1 = \delta$

and $r_1(\delta) = r_1(\alpha_0 + \alpha_1) = \alpha_0 + 2\alpha_1 - \alpha_1 = \alpha_0 + \alpha_1 = \delta$.

__Th__: $W \cdot \{\alpha_0, \alpha_1\} = \{n\delta \pm \alpha_1 \mid n \in \mathbb{Z}\} = \Phi_{real}$

__Def.__ $\Phi_{im} = \{m\delta \mid 0 \neq m \in \mathbb{Z}\}$ is called the set of imaginary roots of $\hat{g}$.

This is a new aspect of KM algebra root systems.

The same construction (untwisted) $\hat{g} = g \otimes_{\mathbb{C}} \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ works for any fin. dim'l (semi) simple g, and these were classified by Kac, and slightly generalized to include "twisted" affine KM algebras.

The twist is a fin. order Dynkin diagram automorphism, σ, so $|\sigma| = 1, 2$ or 3.

Ex: $A_1^{(1)}$ is the "type" affine A_1 with $|\sigma|=1$ (untwisted) with Cartan matrix $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$.

$A_2^{(2)}$ is the twisted affine A_2 with $|\sigma|=2$ (twisted) with Cartan matrix $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$. Try to use this matrix to get the root system Φ.

Key issues: For the affine KM algebras the $\det(A) = 0$ and a central element means the adjoint rep'n is not faithful.

Problem: Find faithful representations of affine KM algs.

Solutions: ① Abstract constructions using the universal enveloping algebra (developed in Humphreys for fin. dim'l algs.)

② Explicit operator constructions from physics: vertex operators, spinor constructions using Clifford algebras / Weyl algebras.

Benefits: Multiple viewpoints give isomorphic pictures, can get unexpected equations.

Power series identities with deep combinatorial content arise from the representation theory.

<u>Ex</u>. Jacobi Triple Product Identity comes from $A_1^{(1)}$:

$$\prod_{n\geq 1}(1-u^n v^n)(1-u^n v^{n-1})(1-u^{n-1}v^n)=\sum_{m\in\mathbb{Z}}(-1)^m u^{\frac{m(m+1)}{2}} v^{\frac{m(m-1)}{2}}$$

The Jacobi quintuple product identity comes from $A_2^{(2)}$.

The Rogers-Ramanujan identities are:

$$\prod_{m\geq 1}(1-q^{5m-4})^{-1}(1-q^{5m-1})^{-1}=\sum_{n\geq 0}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

$$\prod_{m\geq 1}(1-q^{5m-3})^{-1}(1-q^{5m-2})^{-1}=\sum_{n\geq 0}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

The RR-identities occur in rep'n theory of "level 3" modules for $A_1^{(1)}$ and other places.
Vast generalizations have been found in combinatorics and rep'n theory, still an active research area.

Beyond the affine KM algebras, the next type are called "indefinite" by Kac, but the <u>hyperbolic</u> types have been classified.

<u>Ex</u>: For $\ell = 2$, $A = \begin{bmatrix} 2 & -a \\ -b & 2 \end{bmatrix}$ is hyperbolic for $ab > 4$.

Special cases when $a = b$ are called symmetric.

$A = \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$ has an interesting connection with the Fibonacci numbers, so I call it Fib, the rank 2 Fib. hyperbolic KM alg.

Ex: For $\ell = 3$, the hyperbolic KM alg. $\mathcal{F}$ with $A = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ has Dynkin diagram

$\alpha_1 \quad \alpha_0 \quad \alpha_{-1}$

so it contains the affine $A_1^{(1)}$ and the finite type A_2 as obvious subalgebras.

Its Weyl gp. $W = \langle r_1, r_0, r_{-1} \mid 1 = r_i^2 = (r_1 r_{-1})^2 = (r_0 r_{-1})^3 \rangle$ is the hyperbolic triangle group $T(2,3,\infty)$, $W \cong PGL(2,\mathbb{Z})$, and the root system $\Phi = \left\{ \alpha = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \,\middle|\, a,b,c \in \mathbb{Z},\ \det(\alpha) \geq -1 \right\}$ corresponds to definite even binary quadratic forms for $\det(\alpha) \geq 1$, $f(x,y) = ax^2 + 2bxy + cy^2$.