Representation Theory (summary):

Assume: L is a fin. dim'l semisimple Lie alg. over alg. closed field F with char$(F)=0$, H is a fixed Cartan subalg., Φ is the root system of L w.r.t. H, $\Delta=\{\alpha_1,\dots,\alpha_\ell\}$ a base of Φ, W = Weyl group.

For V a fin. dim'l L-module, H acts diagonally on V, so $V=\coprod_{\lambda\in H^*} V_\lambda$ is the direct sum of "weight spaces," $0\neq V_\lambda=\{v\in V \mid h\cdot v=\lambda(h)v, \forall h\in H\}$ (defined for all $\lambda\in H^*$, but only called weight spaces when $V_\lambda\neq 0$).

Let $\Pi(V)=\{\lambda\in H^* \mid V_\lambda\neq 0\}$ be the <u>weights</u> of V.

Ex. $V = L$, the adjoint rep'n (L-module)

$L = H \oplus \coprod_{\alpha \in \Phi} L_\alpha$, $H = L_0$, so $\Pi(L) = \Phi \cup \{0\}$.

Ex. $L = \mathfrak{sl}(2, F)$, $V = V(m)$ irred. with $\dim(V(m)) = m+1$ and $\Pi(V(m)) = \{m, m-2, \ldots, -(m-2), -m\} = \{m - 2i \mid 0 \le i \le m\}$ for $0 \le m \in \mathbb{Z}$.

Note: If $\dim(V) = \infty$ then V may not be the sum of its weight spaces, but $V' = \coprod V_\lambda \le V$ is a direct sum and an L-submod. of V since $L_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}$. **Pf.** $\forall x \in L_\alpha$, $\forall v \in V_\lambda$, $\forall h \in H$,
$h \cdot (x \cdot v) = x \cdot (h \cdot v) + [h, x] \cdot v = x \cdot (\lambda(h) v) + \alpha(h) x \cdot v$
$= (\lambda(h) + \alpha(h))(x \cdot v) = (\lambda + \alpha)(h)(x \cdot v)$ so $x \cdot v \in V_{\lambda+\alpha}$. $\square$

Lemma. Let V be any L-module. Then

(a) $L_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}$, $\forall \alpha \in \Phi$, $\forall \lambda \in H^*$,

(b) $V' = \sum_{\lambda \in H^*} V_\lambda$ is a direct sum and $V' \underset{(L\text{ submod})}{\leq} V$,

(c) If $\dim(V) < \infty$ then $V = V'$.

Standard cyclic modules:

Def. Say that $0 \neq v^+ \in V_\lambda$ is a **maximal vector** of weight λ in an L-module V when $L_\alpha \cdot v^+ = 0$, $\forall \alpha \in \Phi^+$ ($\alpha \in \Delta$ suffices).

Ex. For $V = L$ a simple Lie alg. with max. root $\beta \in \Phi$ w.r.t. Δ, then any $0 \neq x \in L_\beta$ is a max. vector in the adjoint L-module.

For $\dim(V)=\infty$ a max. vector need not exist. For $\dim(V)<\infty$, the Borel subalg. $B(\Delta) = H \oplus \coprod_{\alpha>0} L_\alpha$ is solvable, so has a common eigenvector $v^+ \in V$, by Lie's Th, s.t. $L_\alpha \cdot v^+ = 0$, $\forall \alpha \in \Phi^+$.

Humphreys defines a class of L-modules "generated" by a max. vector, called "standard cyclic" of some weight λ. He uses the "universal enveloping algebra", $\mathcal{U}(L)$, which is given in Section 17. Here I will give a brief summary of $\mathcal{U}(L)$ sufficient to allow us to discuss this topic.

The tensor algebra $\mathcal{T}(V)$ of any vector space V

Def. Let V_i be any vector spaces over field F, $i \in I = \{1, 2, \dots, m\}$. The tensor product $V_1 \otimes \cdots \otimes V_m$ is the unique vector space s.t. for any multilinear map $f : V_1 \times \cdots \times V_m \to W$ $\exists\ \bar{f} : V_1 \otimes \cdots \otimes V_m \to W$, linear such that diagram

$$\begin{tikzcd} V_1 \times \cdots \times V_m \arrow[r, "f"] \arrow[d, "i"] & W \\ V_1 \otimes \cdots \otimes V_m \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

commutes, where $\forall v_i \in V_i$, $i(v_1, \dots, v_m) = v_1 \otimes \cdots \otimes v_m$ and in $V_1 \otimes \cdots \otimes V_m$ we have

① $\forall a \in F$, $a \cdot (v_1 \otimes \cdots \otimes v_m) = v_1 \otimes \cdots \otimes (a v_i) \otimes \cdots \otimes v_m$

② $v_1 \otimes \cdots \otimes (v_i + v_i') \otimes \cdots \otimes v_m = v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_m + v_1 \otimes \cdots \otimes v_i' \otimes \cdots \otimes v_m$.

<u>Note</u>. For $\dim(V_i) = d_i < \infty$ and bases $B_i = \{v_{i1}, \dots, v_{id_i}\}$ for V_i, $1 \le i \le m$, a basis of $V_1 \otimes \cdots \otimes V_m$ is $B = \{v_{1j_1} \otimes v_{2j_2} \otimes \cdots \otimes v_{mj_m} \mid 1 \le j_i \le d_i\}$ consisting of $d_1 d_2 \cdots d_m$ vectors (basic tensors) so $\dim(V_1 \otimes \cdots \otimes V_m) = d_1 \cdots d_m = \prod_{i=1}^{m} \dim(V_i)$.

<u>Special Cases</u>: Let $V_i = V$ be a fixed vector space for all i, and define <u>tensor powers</u> $T^0(V) = F$, $T^1(V) = V$, $T^2(V) = V \otimes V, \dots,$
$T^m(V) = V \otimes V \otimes \cdots \otimes V$ (m factors).
Recursively, $T^0(V) = F$, $T^{m+1}(V) = T^m(V) \otimes V$.

<u>Def</u>. Let $\mathcal{T}(V) = \coprod_{m=0}^{\infty} T^m(V)$ and give it an associative product $(v_1 \otimes \cdots \otimes v_i) \cdot (w_1 \otimes \cdots \otimes w_j) = v_1 \otimes \cdots \otimes v_i \otimes w_1 \otimes \cdots \otimes w_j \in T^{i+j}(V)$ induced by concatenation. This makes $\mathcal{T}(V)$ an assoc. alg. with unit $1 \in F = T^0(V)$ where we understand $(a \cdot 1) \cdot (v_1 \otimes \cdots \otimes v_i) = a(v_1 \otimes \cdots \otimes v_i) = (av_1) \otimes \cdots \otimes v_i = \cdots = v_1 \otimes \cdots \otimes av_i$. As an assoc. alg., $\mathcal{T}(V)$ is generated by 1 and any basis of V. Call $\mathcal{T}(V)$ the <u>tensor algebra</u> of V. It satisfies a universal mapping property: If A is any assoc. alg./F with unit 1, and $\phi : V \to A$ is any F-lin. map, then $\exists!$ F-alg. hom. $\Psi : \mathcal{T}(V) \to A$ s.t.

$\psi(1)=1$ and the diagram

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "\phi"'] & \mathcal{T}(V) \arrow[dl, "\psi"] \\ A & \end{tikzcd}$$

commutes.

Various important assoc. algebras can be constructed as quotients of $\mathcal{T}(V)$.

Def. Let I_s be the 2-sided ideal of $\mathcal{T}(V)$ gen. by $\{x\otimes y - y\otimes x \mid x, y \in V\}$. Then $\mathcal{S}(V) = \mathcal{T}(V)/I_s$ is called the _symmetric algebra_ of V. Since the generators of I_s are in $T^2(V)$, we have $I_s = \bigoplus_{m=2}^{\infty} (I_s \cap T^m(V))$ and the canonical projection map $\sigma : \mathcal{T}(V) \to \mathcal{S}(V)$ is injective on $T^0(V) \oplus T^1(V) = F + V$. So $\mathcal{S}(V)$ contains $\sigma(F\oplus V) \cong F \oplus V$ and $\mathcal{S}(V)$ inherits a $\mathbb{Z}$-grading from $\mathcal{T}(V)$, $\mathcal{S}(V) = \coprod_{i=0}^{\infty} S^i(V)$.

The effect of quotienting by I_s is to make the elements (generators) of V commute in $\mathcal{S}(V)$, which is isomorphic to the polynomial algebra over F in variables $v_1, \dots, v_d$ if $\{v_1, \dots, v_d\}$ is a basis of V.

<u>Def</u>. Let I_a be the 2-sided ideal of $\mathcal{T}(V)$ gen. by $\{x \otimes y + y \otimes x \mid x, y \in V\}$. Then $\mathcal{A}(V) = \mathcal{T}(V)/I_a$ is called the <u>exterior algebra</u> of V. It is similar to $\mathcal{S}(V)$ but where generators <u>anticommute</u>.

A common notation for $\mathcal{A}(V)$ is $\Lambda V = \coprod_{i=0}^{\infty} \Lambda^i V$

but if $\dim(V) = d < \infty$ then $\Lambda^i V = 0$ for $i > d$.

A basis for $\Lambda^i(V)$ is $\{v_{j_1} \wedge \dots \wedge v_{j_i} \mid 1 \leq j_1 < \dots < j_i \leq d\}$

so $\dim(\Lambda^i(V)) = \binom{d}{i}$ and $\dim(\Lambda V) = 2^d$.

Def. Let V be equipped with a bilinear form $(\cdot,\cdot): V \times V \to F$ which is <u>symmetric</u>. Let I_c be the 2-sided ideal of $\mathfrak{T}(V)$ generated by $\{x \otimes y + y \otimes x - (x,y)1 \mid x, y \in V\}$. Then $\mathrm{Cliff}(V,(\cdot,\cdot)) = \mathfrak{T}(V)/I_c$ is the <u>Clifford algebra</u> generated by V and $(\cdot,\cdot)$. Facts about Cliff_m that we discussed recently can be proved from this construction.

Exercise. How would you get the Weyl algebra from this type of construction assuming V has an <u>anti-symm.</u> bil. form $(\cdot,\cdot)$?