<u>Def</u>. Let L be any Lie alg. and let $\mathcal{T}(L)$ be the tensor alg. of L as a vector space. Let I_L be the 2-sided ideal of $\mathcal{T}(L)$ gen. by $\{x \otimes y - y \otimes x - [x,y] \mid x, y \in L\}$. We call $\mathcal{T}(L)/I_L$ the <u>universal enveloping algebra</u> of L and denote it by $\mathcal{U}(L)$. $\pi : \mathcal{T}(L) \to \mathcal{U}(L)$ is the canonical projection map. $I_L \subseteq \coprod_{i>0} T^i(L)$ so

$$\pi(T^0(L)) = \pi(F) \cong F \subseteq \mathcal{U}(L).$$

It can be shown that $\pi(T^1(L)) = \pi(L) \cong L \subseteq \mathcal{U}(L)$ so $\mathcal{U}(L)$ is the assoc. alg. gen. by L with relations $xy - yx = [x,y]$, $\forall x, y \in L$.

In $\mathcal{U}(L)$ expressions $x^n = \underbrace{xx \cdots x}_{n\text{-times}}$ make sense.

Def. If V is an L-module, say V is <u>standard cyclic</u> with max. weight λ if $V = \mathcal{U}(L)\cdot v^+$ for a max. vector v^+ of wt. λ.

Note: Any L-module is a $\mathcal{U}(L)$-module, and conversely.

Notation. Fix $0 \neq x_\alpha \in L_\alpha$ for each pos. root $\alpha \in \Phi^+$, then choose $y_\alpha \in L_{-\alpha}$ s.t. $[x_\alpha, y_\alpha] = h_\alpha \in H$ and $\{x_\alpha, y_\alpha, h_\alpha\}$ spans a subalg. isomorphic to $\mathfrak{sl}(2,F)$.
$\forall\, \lambda, \mu \in H^*$, recall the partial ordering $\lambda \geq \mu$ iff

$$\lambda - \mu = \sum_{i=1}^{\ell} k_i \alpha_i \quad \text{for } 0 \leq k_i \in \mathbb{Z}$$

$$= \sum_{\alpha > 0} k_\alpha \alpha \quad \text{for } 0 \leq k_\alpha \in \mathbb{Z}.$$

<u>Th</u>. Let V be a std. cyclic L-module with max. vector $v^+ \in V_\lambda$. Let $\Phi^+ = \{\beta_1, \dots, \beta_m\}$. Then

(a) V is spanned by $\{ y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m} \cdot v^+ \mid 0 \le i_j \in \mathbb{Z} \}$ and so $V = \coprod V_\mu$ is the direct sum of its wt. spaces.

(b) $\Pi(V) \subseteq \{ \mu = \lambda - \sum_{i=1}^{\ell} k_i \alpha_i \mid 0 \le k_i \in \mathbb{Z} \}$ so $\mu \in \Pi(V)$ implies $\mu \le \lambda$.

(c) $\forall \mu \in H^*$, $\dim(V_\mu) < \infty$ and $\dim(V_\lambda) = 1$.

(d) If $W \le V$ is an L-submod then $W = \coprod W_\mu$.

(e) V is indecomposable with a unique max. proper submod. W_{max} s.t. V/W_{max} is irred.

(f) Every non-zero homom. image of V is also a std. cyclic L-mod with max. wt. λ.

Pf. $L = H \oplus N^- \oplus N^+$ where $N^{\pm} = \coprod_{\alpha \in \Phi^{\pm}} L_\alpha$.

The Poincaré-Birkhoff-Witt (PBW) Theorem is a structure result for $\mathcal{U}(L) = \mathcal{U}(N^-)\mathcal{U}(H)\mathcal{U}(N^+)$ giving a basis for each factor. (See Section 17.) This implies that $V = \mathcal{U}(L)\cdot v^+ = \mathcal{U}(N^-)\mathcal{U}(H)\mathcal{U}(N^+)\cdot v^+ = \mathcal{U}(N^-)\cdot v^+$ since $\mathcal{U}(H)\,\mathcal{U}(N^+)\cdot v^+ = F v^+$. Since a PBW basis of $\mathcal{U}(N^-)$ is $\{\underbrace{y_{\beta_1}^{i_1}\cdots y_{\beta_m}^{i_m}}_{\text{"}\,\cdot v^+} \mid 0 \le i_j \in \mathbb{Z}\}$, we get (a). The weight of $\text{"}\cdot v^+$ is $\lambda - \sum_{j=1}^{m} i_j \beta_j = \mu$ finishes (a) and (b). For a fixed $\mu \le \lambda$ there are only finitely many choices of i_j, and for $\mu = \lambda$, there is only one choice, proving (c).

(d) Let $W \leq V$ be an L-submod. and write any $w \in W$ as $w = \sum_{i=1}^{n} v_i$ for $v_i \in V_{\mu_i}$ for distinct weights $\mu_i \in \Pi(V)$. We want to show each $v_i \in W$. If not, choose $w \in W$ s.t. $n > 1$ is minimal with none of the v_i in W. Since $\mu_1 \neq \mu_2$, $\exists h \in H$ s.t. $\mu_1(h) \neq \mu_2(h)$. Then $h \cdot w = \sum_{i=1}^{n} \mu_i(h) v_i \in W$ and $\mu_1(h) w = \sum_{i=1}^{n} \mu_1(h) v_i \in W$ so the difference

$$(h - \mu_1(h)\mathbb{1}) \cdot w = \sum_{i=1}^{n} (\mu_i(h) - \mu_1(h)) v_i = \sum_{i=2}^{n} (\mu_i(h) - \mu_1(h)) v_i$$

$\in W$ is non-zero in W since $\mu_2(h) \neq \mu_1(h)$, but with fewer terms than the minimal n. This contradiction proves the result.

(e) If $W \lneq V$ then $W = \coprod_{\mu \neq \lambda} W_\mu$ cannot include V_λ (otherwise $W = V$). The sum of all such proper

submodules is the unique max. proper submod W_{max}, and V/W_{max} must be irreducible.
If $V = W_1 \oplus W_2$ (decomposable) with $W_i \subsetneq V$ then $W_i \subseteq W_{max}$ so $W_1 + W_2 \subseteq W_{max} \subsetneq V$, contradiction.
(f) If $\phi : V \to U$ is an L-mod. map with $\mathrm{Im}(\phi) \neq 0$ then $\mathrm{Im}(\phi) \cong V/\mathrm{Ker}(\phi)$ is a quotient of V by a proper submod, so $\phi(v^+)$ is a max. vector of wt. λ in $\mathrm{Im}(\phi)$. $\square$

<u>Cor</u>. Let V be as in the last Theorem with the additional assumption that V is an irred. L-mod. Then Fv^+ is unique highest weight space V_λ in V.
<u>Pf</u>. If w^+ is another max. vector of some wt. μ, then $V = \mathfrak{U}(L)\cdot w^+$ so $\lambda \leq \mu$ and $\mu \leq \lambda$ so $\lambda = \mu$. Since $\dim(V_\lambda) = 1$, $Fv^+ = Fw^+ = V_\lambda$. $\square$

<u>Th</u>. Let V, W be std. cyclic L-modules of highest wt. $\lambda \in H^*$. If V and W are irred. then $V \cong W$.

<u>Pf</u>. Let $X = V \oplus W$ and suppose $v^+ \in V_\lambda, w^+ \in W_\lambda$ are max. vectors, so $x^+ = (v^+, w^+) \in X_\lambda$ is a max. vector. Let $Y \subseteq X$ be the std. cyclic L-mod in X gen. by x^+, $Y = \mathcal{U}(L) \cdot x^+$. Define the projection maps (L-mod.) $\rho : Y \to V$ and $\rho' : Y \to W$ onto the two components. Since $\rho(x^+) = v^+$ and $\rho'(x^+) = w^+$, $\rho(\mathcal{U}(L) \cdot x^+) = \mathcal{U}(L)\rho(x^+) = \mathcal{U}(L) v^+ = V$ and $\rho'(\mathcal{U}(L) x^+) = W$ so $\rho(Y) = V$ and $\rho'(Y) = W$ are their images, both irred. so both are isom. to quotient, Y/Y_{max} of Y by its unique max. proper submodule, Y_{max}. $\square$

Constructions of std. cyclic L-modules.

<u>As an induced module</u>: Let $D_\lambda = Fv^+$ (1-dim'l) with action of $B = B(\Delta) = H + N^+$ by $\left(h + \sum_{\alpha > 0} x_\alpha\right) \cdot v^+ = h \cdot v^+ = \lambda(h)\, v^+$ for given fixed $\lambda \in H^*$.

Then D_λ is a B-mod. so it is a $\mathcal{U}(B)$-module.

Define $Z(\lambda) = \mathcal{U}(L) \otimes_{\mathcal{U}(B)} D_\lambda$ which is a $\mathcal{U}(L)$-mod. under left mult. by $\mathcal{U}(L)$.

<u>Th</u>. $Z(\lambda)$ is a std. cyclic L-mod. of wt. λ.

<u>Pf</u>. $Z(L)$ is generated by $1 \otimes v^+$, which is a max. vector of wt. λ. $\mathcal{U}(L) = \mathcal{U}(N^-)\,\mathcal{U}(B)$ so $\mathcal{U}(L) \cdot (1 \otimes v^+) = \mathcal{U}(N^-)\,\mathcal{U}(B) \cdot 1 \otimes v^+ = \mathcal{U}(N^-) \otimes \mathcal{U}(B) v^+ = \mathcal{U}(N^-) \otimes Fv^+$ is spanned by $\{ y_{\beta_1}^{i_1} \cdots y_{\beta_m}^{i_m}\, 1 \otimes v^+ \mid 0 \le i_j \in \mathbb{Z} \}$ and these are indep. in $\mathcal{U}(N^-)$.

As a $\mathcal{U}(N^-)$-mod., $Z(\lambda) \cong \mathcal{U}(N^-)$.

<u>Construction of $Z(\lambda)$ by generators & relations:</u>

Let $I(\lambda)$ be the left ideal of $\mathcal{U}(L)$ gen. by $\{X_\beta, h_\alpha - \lambda(h_\alpha)1 \mid \alpha \in \Phi, \beta \in \Phi^+\}$. These elts. annihilate v^+, so $I(\lambda)\cdot v^+ = 0$, and get canonical map $\pi_\lambda : \mathcal{U}(L)/I(\lambda) \xrightarrow{\text{onto}} Z(\lambda)$ s.t. $\pi_\lambda(1+I(\lambda)) = v^+$.
Also $\pi_\lambda(\mathcal{U}(B)/I(\lambda)) = Fv^+$. Using PBW basis of $\mathcal{U}(L)$, can see π_λ is injective as well, so $Z(\lambda) \cong \mathcal{U}(L)/I(\lambda)$.

<u>Th.</u> For any $\lambda \in H^*$, $\exists$ irred std. cyclic L-module $V(\lambda)$ of highest weight λ.

<u>Pf.</u> Get $V(\lambda) = Z(\lambda)/Z(\lambda)_{max}$ quotient of $Z(\lambda)$ by its max. proper submod. $\square$