Goals: ① Find those $\lambda \in H^*$ s.t. $\dim(V(\lambda)) < \infty$

② For $\dim(V(\lambda)) < \infty$, $\mu \in \Pi(V(\lambda))$, find $\dim(V(\lambda)_\mu) = \mathrm{Mult}_\lambda(\mu)$.

If V is a fin. dim'l irred. L-mod, then V has some max. vector of wt. λ uniquely det'd by $V = \mathcal{U}(L) \cdot v^+ = V(\lambda)$.

$\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. For each $1 \le i \le \ell$ let $S_i \cong \mathfrak{sl}(2,F)$ be the subalg. with basis $x_{\alpha_i} \in L_{\alpha_i}$, $y_{\alpha_i} \in L_{-\alpha_i}$, $h_{\alpha_i} \in H$. $V = V(\lambda)$ is an S_i-module (fin. dim'l) so V has a decomposition into a direct sum of irred. S_i-modules. Use $\mathfrak{sl}(2,F)$ rep'n theory. $v^+ \in V(\lambda)$ max. vector for L is also max. for S_i. So $h_i = h_{\alpha_i}$ acts on v^+ as scalar $\lambda(h_i) \in \mathbb{Z}_{\ge 0}$.

__Th__. If V is an irred. fin. dim'l L-mod of highest wt. λ, then $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for $1 \leq i \leq \ell$. Also, $\forall \mu \in \Pi(V)$, $\mu(h_i) = \langle \mu, \alpha_i \rangle \in \mathbb{Z}$ for $1 \leq i \leq \ell$, so $\Pi(V) \subseteq \Lambda$, weight lattice of L, and λ is a dominant weight.

__Def__. $\forall \mu \in H^*$, call μ a weight of L-module V when $V_\mu \neq 0$. If $\mu(h_i) = \langle \mu, \alpha_i \rangle \in \mathbb{Z}$, call μ an __integral weight__, and Λ is the lattice of all integral weights. If $0 \leq \langle \mu, \alpha_i \rangle \in \mathbb{Z}$, call μ __dominant__ integral, and $\Lambda^+ = \{\mu \in H^* \mid 0 \leq \langle \mu, \alpha_i \rangle \in \mathbb{Z}\}$
$\Lambda = \mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_\ell$ for __fundamental__ weights λ_i s.t. $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq \ell$. Root lattice $\Lambda_r \subseteq \Lambda$
$\Lambda_r = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_\ell$. Let $\Pi(\lambda) = \Pi(V(\lambda))$.

<u>Th.</u> If $\lambda \in \Lambda^+$ then irred. L-mod $V(\lambda)$ is fin. dim'l, $\Pi(\lambda)$ is permuted by Weyl gp. W, and $\dim(V_\mu) = \dim(V_{\sigma\mu})$, $\forall \sigma \in W$.

<u>Cor.</u> Map $\lambda \mapsto V(\lambda)$ is a bijection between Λ^+ and isom. classes of fin. dim'l irred. L-modules.

<u>Pf. of Th.</u> First we prove this Lemma in $\mathcal{U}(L)$:

<u>Lemma.</u> In $\mathcal{U}(L)$ we have for $0 \le k \in \mathbb{Z}$, $1 \le i, j \le \ell$,

(a) $[x_j, y_i^{k+1}] = 0$ for $i \ne j$,

(b) $[h_j, y_i^{k+1}] = -(k+1)\,\alpha_i(h_j)\, y_i^{k+1}$,

(c) $[x_i, y_i^{k+1}] = -(k+1)\, y_i^k (k \cdot 1 - h_i)$.

<u>Pf.</u> (a) For $i \ne j$, $[x_j, y_i] \in L_{\alpha_j - \alpha_i} = \{0\}$ since $\alpha_j - \alpha_i \notin \Phi$.

Since x_j commutes with y_i, it commutes with any power of y_i.

(b) By induction on k with base case $[h_j, y_i] = -\alpha_i(h_j) y_i$ for $k=0$. Generally, $h_j y_i^{k+1} - y_i^{k+1} h_j =$

$$(h_j y_i^k - y_i^k h_j) y_i + y_i^k (h_j y_i - y_i h_j)$$

$$= -k \alpha_i(h_j) y_i^k y_i + y_i^k(-\alpha_i(h_j) y_i) = -(k+1)\alpha_i(h_j) y_i^{k+1}.$$

(c) $[x_i, y_i^{k+1}] = x_i y_i^{k+1} - y_i^{k+1} x_i = [x_i, y_i] y_i^k + y_i [x_i, y_i^k]$

$$= h_i y_i^k + y_i [x_i, y_i^k] = h_i y_i^k - k y_i^k ((k-1) \cdot 1 - h_i)$$

$$\overset{(b)}{=} y_i^k h_i - k 2 y_i^k - k y_i^k ((k-1) \cdot 1 - h_i)$$

$$= y_i^k (h_i - 2k - k(k-1) + k h_i) = y_i^k ((k+1) h_i - k(k+1))$$ gives (c). $\square$

Pf. of Th.: Let $\phi: L \to gl(V)$ be the rep'n map from L-mod. V. Let $v^+ \in V_\lambda$ be a max. vector, and $m_i = \lambda(h_i)$ for $1 \leq i \leq l$, so $0 \leq m_i \in \mathbb{Z}$.

Step(1): $y_i^{m_i+1} \cdot v^+ = 0$. Pf. Let $w = y_i^{m_i+1} \cdot v^+$. By Lemma (a), for $i \neq j$ have $x_j \cdot w = 0$. For $i = j$, have

$x_i \cdot w = x_i(y_i^{m_i+1} \cdot v^+) = y_i^{m_i+1} \cdot (x_i \cdot v^+) - (m_i+1) y_i^{m_i}(m_i - h_i) v^+$
$= 0 - (m_i+1) y_i^{m_i}(m_i - \lambda(h_i)) v^+ = 0$. If $w \neq 0$ it would be a max. vector in V of wt. $\lambda - (m_i+1)\alpha_i \neq \lambda$.

Step(2): For $1 \leq i \leq l$, V contains a non-zero fin. dim'l S_i-mod. Pf. The span of $\{v^+, y_i \cdot v^+, \ldots, y_i^{m_i} \cdot v^+\}$ is y_i-stable by Step (1). It is h_i-stable since each vector is in a wt. space of V, V_μ, $\mu = \lambda - j\alpha_i$. It is x_i-stable by Lemma (c).

<u>Step(3)</u>: V is the sum of fin. dim'l S_i-submods!

<u>Pf</u>. Let V' be the sum of all fin. dim'l S_i-submods of V. Step(2) says $V' \neq 0$. Let W be any f.d. S_i-submod of V. The span of $\{X_\alpha \cdot W \mid \alpha \in \Phi\}$ is f.d. and is S_i-stable, so is in V', so V' is L-stable. As a nonzero L-submodule of V, (irred) $V' = V$.

<u>Step(4)</u>: For $1 \leq i \leq \ell$, $\phi(X_i)$ and $\phi(Y_i)$ are locally nilp. endo's of V. <u>Pf</u>. $\forall v \in V$, v is in a fin. sum of f.d. S_i-submod's (a f.d. S_i-submod.) by Step(3). On such modules $\phi(X_i)$ and $\phi(Y_i)$ are nilp.

<u>Step (5)</u>: $s_i = \exp(\phi(X_i)) \exp(\phi(-Y_i)) \exp(\phi(X_i))$ is an autom. of V. (See Humphreys, p. 33.)

<u>Step (6)</u>: $\forall \mu \in \Pi(V)$, $s_i(V_\mu) = V_{\sigma_i(\mu)}$ for simple reflection $\sigma_i \in W$.

<u>Pf</u>. V_μ is in a f.d. S_i-submod W in V' from Step (3) and $s_i|_W$ is the autom. called τ on p. 33, where this was shown for $\mathfrak{sl}(2,F)$-modules.

<u>Step (7)</u>: $\Pi(\lambda)$ is W-stable and $\dim(V_\mu) = \dim(V_{\sigma\mu})$ $\forall \mu \in \Pi(\lambda)$, $\sigma \in W$. <u>Pf</u>. Follows from Step (6) and $W = \langle \sigma_1, \dots, \sigma_\ell \rangle$.

<u>Step (8)</u>: $\Pi(\lambda)$ is finite. <u>Pf</u>. We know that $W \cdot \{\mu \in \Lambda^+ \mid \mu \leq \lambda\}$ is finite. By Th. on p. 325 of these notes, and Step (7), $\Pi(\lambda)$ is contained in that finite set.

Step(9): $\dim(V)<\infty$. Pf. $\dim(V_\mu)<\infty$, $\forall \mu\in\Pi(\lambda)$, and $\Pi(\lambda)$ finite gives the result. $\square$

Let $V=V(\lambda)$ for $\lambda\in\Lambda^+$, and let $\mu\in\Pi(\lambda)$ and $\alpha\in\Phi$.

Let $W=\sum_{i\in\mathbb{Z}} V_{\mu+i\alpha} \leq V$ (subspace), so W is S_α-invariant.

By Weyl's complete reducibility Thm (Sec. 7), we have $\{\mu+i\alpha\in\Pi(\lambda)\mid -r\leq i\leq q\}$ form an unbroken "α-string through μ". Also, $\sigma_\alpha\in W$ flips the string, and $r-q=\langle\mu,\alpha\rangle$.

$\Pi(\lambda)$ is saturated (See Section 13.4),

Prop. If $\lambda\in\Lambda^+$, so $\mu\in\Pi(\lambda)$ iff $\sigma\mu\leq\lambda$, $\forall\sigma\in W$.

Ex: See $\Pi(\lambda)$ for $\lambda=4\lambda_1+3\lambda_2$ and L type A_2 on page 115 of Humphreys.