Homework #1 (Humphreys): Pages 5-6, Problems 4, 6, 11, 12.

Facts about ideals in a Lie algebra.

① If $I, J \trianglelefteq L$ are ideals, then

$I+J=\{x+y\in L \mid x\in I, y\in J\} \trianglelefteq L$ and

$[I,J]=\text{span}\{[x,y]\in L \mid x\in I, y\in J\} \trianglelefteq L.$

Note: $[I,J] \trianglelefteq I\cap J \trianglelefteq L.$

__Def.__ For $K\leq L$ (subspace of L) define the __normalizer__ of K in L to be

$N_L(K)=\{x\in L \mid [x,K]\subseteq K\}$. Then $N_L(K)\leq L$ is a subalgebra.

Note: If K is a subalgebra, so $[K,K] \subseteq K$, then $K \trianglelefteq N_L(K)$ and $N_L(K)$ is the largest subalg. of L in which K is an ideal.

<u>Def</u>. Say K is self-normalizing if $K = N_L(K)$.

<u>Def</u>. For subset $X \subseteq L$, $C_L(X) = \{y \in L \mid [y, X] = 0\}$ is the <u>centralizer</u> of X in L. This is a subalgebra of L, and $C_L(L) = Z(L)$ center.

<u>Prop</u>. If $\phi : L \to E$ is a Lie alg. hom. then $L/\mathrm{Ker}\,\phi \cong \mathrm{Im}(\phi)$. If $I \trianglelefteq L$ with $I \leq \mathrm{Ker}\,\phi$ then $\exists\, \psi : L/I \to E$ s.t. $\phi = \psi \circ \pi$ where $\pi : L \to L/I$ is "canonical" projection $\pi(x) = x + I$.

$$\begin{tikzcd} L \arrow[r, "\phi"] \arrow[d, "\pi"] & E \\ L/I \arrow[ur, "\psi"'] & \end{tikzcd}$$

For what $I \trianglelefteq L$ does such a hom. ψ exist? Must have $\psi(x+I) = \phi(x)$ a well-defined map. If $x+I = y+I$ for $x, y \in L$, then need $\phi(x) = \phi(y)$. Only know that $x-y \in I$. Must imply $\phi(x-y) = \phi(x) - \phi(y) = 0$, so need to have $x-y \in I \Rightarrow x-y \in \mathrm{Ker}(\phi)$. This is only guaranteed when $I \leq \mathrm{Ker}(\phi)$.

When $I = \mathrm{Ker}(\phi)$ then ψ is injective since $\psi(x+I) = \psi(y+I)$ gives $\phi(x) = \phi(y)$ so $\phi(x-y) = 0$ so $x-y \in \mathrm{Ker}(\phi)$ so $x+I = y+I$.

ψ is a Lie alg. hom. since $\forall x, y \in L$,
$\psi[x+I, y+I] = \psi([x,y]+I) = \phi([x,y]) = [\phi(x), \phi(y)]$
$= [\psi(x+I), \psi(y+I)]$. $\square$ (1st Isom. Thm.)

<u>Th</u> (2nd Isom.): If $I, J \trianglelefteq L$ and $I \subseteq J$
then $J/I \trianglelefteq L/I$ and $(L/I)/(J/I) \cong L/J$.

<u>Th</u> (3rd Isom.) If $I, J \trianglelefteq L$ then
$(I+J)/J \cong I/(I \cap J)$.

These have standard proofs just like the corresponding proofs in ring theory.

Note: $\operatorname{Ker}(ad) = \{x \in L \mid ad_x = 0\} =$
$\{x \in L \mid 0 = ad_x(y) = [x,y], \forall y \in L\} = Z(L)$.
If L is simple then $Z(L) \trianglelefteq L$ and
$Z(L) \neq L$ (otherwise $[L,L] = 0$) so $Z(L) = 0$
so $ad: L \to gl(L)$ is injective. It means
any simple Lie algebra is isomorphic to a "linear Lie algebra" (Lie subalgebra of $gl(V)$ for some V).

__Def.__ $\operatorname{Aut}(L) = \{\phi: L \to L \mid \phi$ is an isomorphism$\}$
This is a group. EX.: If $L \leq gl(V)$ and $g \in GL(V)$ and $gLg^{-1} = L$ then $g \in \operatorname{Aut}(L)$.

$GL(V) = \{f : V \to V \mid f \text{ is invertible linear map}\}$ acts by conjugation on any $X \in End(V)$ and respects associative composition product:

$$g(X \circ Y)g^{-1} = (gXg^{-1}) \circ (gYg^{-1})$$ so also respects commutator "Lie bracket":

$$g[X,Y]g^{-1} = g(X \circ Y - Y \circ X)g^{-1} = (gXg^{-1})(gYg^{-1}) - (gYg^{-1})(gXg^{-1})$$
$$= [gXg^{-1}, gYg^{-1}].$$

<u>Ex</u>: $L = \mathfrak{sl}(n,F)$, $\forall g \in GL(n,F)$, $\forall X \in L$,

$Tr(gXg^{-1}) = Tr((Xg^{-1})g) = Tr(X) = 0$ so $gXg^{-1} \in L$.

Def. Say $x \in L$ is <u>ad-nilpotent</u> when $ad_x \in End(L)$ is nilpotent, that is, $(ad_x)^k = 0$ for some $0 < k \in \mathbb{Z}$. It means $\forall y \in L$, $(ad_x)^k(y) = \underbrace{[x,[x,[\cdots[x,}_{k\text{-times}} y]\cdots]]] = 0$.

Def. $\exp(ad_x) = \sum_{n=0}^{k-1} \frac{(ad_x)^n}{n!} = 1 + ad_x + \frac{(ad_x)^2}{2!} + \cdots + \frac{(ad_x)^{k-1}}{(k-1)!}$ is a finite sum for $(ad_x)^k = 0$.
Need $char(F) = 0$ for $\frac{1}{n!}$ factors.

Recall; $ad_x \in Der(L) = \{\delta : L \to L \mid \delta \text{ is a derivation of } L\}$

Th. (Leibniz Rule) Let $(A, *)$ be any assoc. algebra and let $\delta \in Der(A)$. Then

$$\frac{\delta^n}{n!}(x*y) = \sum_{i=0}^{n} \frac{\delta^i(x)}{i!} * \frac{\delta^{n-i}(y)}{(n-i)!} \quad \text{for any } 0 \leq n \in \mathbb{Z}.$$

Pf. Exercise. $\delta^0 = Id_A$. Case $n=1$ is definition of derivation. Proof by induction on n. $\square$

Cor. For nilpotent $\delta \in Der(A)$, say $\delta^k = 0$, $\forall x, y \in A$, $\exp(\delta)(x) * \exp(\delta)(y) = \exp(\delta)(x*y)$, so $\exp(\delta) \in Aut A$.

Pf. See Humphreys, page 9.

Check $\exp(\delta)$ is invertible: Write $\exp(\delta) = 1 + \eta$. Then inverse is $1 - \eta + \eta^2 - \eta^3 + \cdots + (-1)^{k-1}\eta^{k-1}$.

__Def.__ For ad_x nilp., say $\exp(ad_x) \in Aut(L)$ is an __inner__ automorphism, and $Int(L) = \langle \exp(ad_x) \mid ad_x \text{ is nilp} \rangle \trianglelefteq Aut(L)$ is a __normal__ subgroup because $\forall\, \phi \in Aut(L)$, $\forall x \in L$, ad-nilp, $\phi \circ (ad_x) \circ \phi^{-1} = ad_{\phi(x)}$ so

$$\phi \circ (ad_x)^n \circ \phi^{-1} = (ad_{\phi(x)})^n \text{ so}$$

$$\phi \circ (\exp(ad_x)) \circ \phi^{-1} = \exp(ad_{\phi(x)}).$$

Let's do some explicit calculations in $\mathfrak{sl}(2,F)$. In Humphreys, notation is

$$x = e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\quad y = f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let $\sigma = \exp(\mathrm{ad}_x)\exp(\mathrm{ad}_{-y})\exp(\mathrm{ad}_x) \in \mathrm{Int}(L)$ #6

Compute: $\exp(\mathrm{ad}_x): x \longmapsto x$

$: y \longmapsto y + \underbrace{[x,y]}_{=h} + \frac{1}{2!}\underbrace{[x,h]}_{=-2x} = y + h - \frac{2x}{2}$

$: h \longmapsto h + \underbrace{[x,h]}_{=-2x} = h - 2x$

use: $[x,y] = h$

$[h,x] = 2x$

$[h,y] = -2y$

$\exp(\mathrm{ad}_{-y}): x \longmapsto x - \underbrace{[y,x]}_{=h} - \frac{[y,h]}{2!} = x + h - y$

$: y \longmapsto y$

$: h \longmapsto h + \underbrace{[-y,h]}_{=-2y} + \frac{1}{2!}\underbrace{[-y,-2y]}_{=0} = h - 2y$

$\exp(\mathrm{ad}_h): x \longmapsto x + \underbrace{[h,x]}_{=2x} + \frac{1}{2!}\underbrace{[h,2x]}_{=4x} + \cdots$ infinite sum! Don't use here.

So $\sigma(x) = \exp(\mathrm{ad}_x)\exp(\mathrm{ad}_{-y})(x)$

$= \exp(\mathrm{ad}_x)(x+h-y) = x+h-2x-(y+h-x) = -y$

$\sigma(y) = \exp(\mathrm{ad}_x)\exp(\mathrm{ad}_{-y})(y+h-x)$

$= \exp(\mathrm{ad}_x)(y+h-2y-(x+h-y))$

$= \exp(\mathrm{ad}_x)(-x) = -x$

$\sigma(h) = \exp(\mathrm{ad}_x)\exp(\mathrm{ad}_{-y})(h-2x)$

$= \exp(\mathrm{ad}_x)(h-2y-2(x+h-y))$

$= \exp(\mathrm{ad}_x)(-h-2x) = -h+2x-2x = -h$

Summary: $\sigma(x) = -y$, $\sigma(y) = -x$, $\sigma(h) = -h$.

So $\sigma^2 = I$, σ is an <u>involution</u>.

Could represent σ as a 3×3 matrix w.r.t. basis $\{x, y, h\}$ of $L = \mathfrak{sl}(2,F)$.

But the matrices in L are 2×2 and we have

$$\exp(x) = \begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}0&1\\0&0\end{bmatrix} = \begin{bmatrix}1&1\\0&1\end{bmatrix} \in SL(2,F),$$

$$\exp(-y) = \begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}0&0\\-1&0\end{bmatrix} = \begin{bmatrix}1&0\\-1&1\end{bmatrix} \in SL(2,F) \quad \text{so}$$

$$s = \exp(x)\exp(-y)\exp(x) = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}$$

$$= \begin{bmatrix}0&1\\-1&1\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$$ and conjugation by s

does: $$sxs^{-1} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}0&0\\0&-1\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}0&0\\-1&0\end{bmatrix} = -y$$

$$sys^{-1} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}0&-1\\0&0\end{bmatrix} = -x$$

and $shs^{-1} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}0&-1\\-1&0\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}-1&0\\0&1\end{bmatrix}$

$= -h$

So the action of σ on $L = \mathfrak{sl}(2,F)$ is the same as conjugation by $s = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$.

This is an example of a general result.

<u>Th.</u> Let $L \leq \mathfrak{gl}(V)$, $\mathrm{char}(F)=0$, $x \in L$ a nilp. endomorphism of V. Then $\forall y \in L$,

$$\exp(x)\, y\, (\exp(x))^{-1} = \exp(\mathrm{ad}_x)(y)$$

<u>Pf.</u> In the assoc. ring $\mathrm{End}(V)$ let λ_x = left mult. by x, ρ_x = right mult. by x so these commute and are nilp.

and $ad_x = \lambda_x + \rho_x$. As commuting operators on V we can use laws of exp. to get $\exp(ad_x) = \exp(\lambda_x + \rho_x) = \exp(\lambda_x)\exp(\rho_x)$

$$= \lambda_{\exp(x)} \cdot \rho_{\exp(-x)}, \text{ so get}$$

$$\exp(ad_x)(y) = (\exp(x))\, y\, (\exp(-x))$$
$$= (\exp(x))\, y\, (\exp(x))^{-1}. \qquad \square$$

Homework #2: (Humphreys) page 10: #5, 11.