The connection between Lie algebras and Lie groups is important, but not a theme of this course. Even so, it can be fun to try calculations in simple cases, and you can read more about it in many textbooks.

__Ex__: For $t \in F$, $\exp(te) = \exp\left(\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \in SL(2,F)$

so $\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & t+s \\ 0 & 1 \end{bmatrix}$ means

$$\left\{\exp\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \,\middle|\, t \in F\right\} \cong F \text{ (under + in F)}$$

one-parameter group under matrix mult.

Similarly, $\exp(tf) = \exp\left(\begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ gives another subgroup of $SL(2,F)$.

What about $\exp(th) = \exp\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}$?

For $0 \le n \in \mathbb{Z}$, $\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}^n = \begin{bmatrix} t^n & 0 \\ 0 & (-t)^n \end{bmatrix}$ so

$$\exp(th) = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} t^n & 0 \\ 0 & (-t)^n \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \in SL(2,F)$$

Here we are assuming $\mathrm{char}(F) = 0$, usually F is $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$, so e^t is well-known in F.

Again, $\exp(th) \cdot \exp(sh) = \exp((t+s)h)$ even though h is <u>not</u> <u>nilpotent</u>.

Let look at $x = e - f$ and try to calculate

$$\exp(tx) = \exp \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}^n .$$

$$(tx)^2 = \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} = \begin{bmatrix} -t^2 & 0 \\ 0 & -t^2 \end{bmatrix} = -t^2 I_2 \quad \text{so}$$

$(tX)^3 = -t^2(tX) = -t^3 X,$

$(tX)^4 = -t^2(tX)^2 = (-t^2)^2 I_2,$

$(tX)^5 = (-t^2)^2(tX) = t^5 X,$ etc.

$$\exp(tX) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(tX)^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(tX)^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-t^2)^n I_2 + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-t^2)^n t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \in SL(2,F)$$ but it is the (compact) circle for $t \in \mathbb{R}$.

<u>Note</u>: For $t \in \mathbb{R}$, $i = \sqrt{-1} \in \mathbb{C}$, we have
$\exp(ith) = \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \in SL(2, \mathbb{C})$ is isomorphic
to the unit circle in $\mathbb{C}$.

For $L = \mathfrak{sl}(n+1, \mathbb{C})$, the abelian subalgebra of diagonal matrices of trace 0 exponentia. to diagonal matrices of det 1 in $SL(n+1, \mathbb{C})$.
Using $\sqrt{-1}(E_{ii} - E_{(i+1),(i+1)})$, $1 \le i \le n$, exp map gives direct product of circles, that is, a <u>torus</u>.

Back to Lie algebras: For Lie algebra L,

Define $L^{(0)} = L$, $L^{(1)} = [L, L]$, $L^{(2)} = [L^{(1)}, L^{(1)}]$, ... $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ for $i \geq 0$. This is called the <u>derived series</u> of L. It is easy to see that each $L^{(i)} \trianglelefteq L$ is an ideal of L.

<u>Def</u>. Say L is <u>solvable</u> if $L^{(n)} = 0$ for some n.

<u>EX</u>: L abelian means $L^{(1)} = 0$, so L is solv.
L simple means $L^{(1)} = L$ so $L^{(i)} = L$ for all i so <u>not</u> solvable.

Ex: The matrix Lie algebra of upper triangular matrices $t(n,F) = \{A = [a_{ij}] \in F^n_n \mid a_{ij} = 0 \text{ if } i > j\}$ is solvable. Use standard basis matrices $\{E_{ij} \mid i \leq j\}$ and commutator formula (page 6). 56

$[E_{ii}, E_{ij}] = E_{ij}$ for $i < j$ shows that $\eta(n,F) \subseteq [L,L]$ for $L = t(n,F)$, and $[d,\eta] = \eta$.
$t(n,F) = d(n,F) \oplus \eta(n,F)$ so $[L,L] = \eta$.
The grading of $t(n,F)$ by level shows that $L^{(i)} = 0$ for $2^{i-1} > n-1$.

Note: For any L, $(L^{(i)})^{(j)} = L^{(i+j)}$.

Th: Let L be a Lie algebra.

(a) If L is solvable, so are all subalgebras and hom. images of L.

(b) If $I \trianglelefteq L$ and I is solvable and L/I is solvable then L is solvable.

(c) If $I, J \trianglelefteq L$ are solvable, so is $I+J$.

Pf. (a) For any subalg. $K \leq L$ it is clear that $K^{(i)} \leq L^{(i)}$ so if $L^{(i)} = 0$ for some i, then $K^{(i)} = 0$ (but $K^{(j)}$ may be 0 for some $j < i$).

If $\phi : L \to M$ is onto, check that $\phi(L^{(i)}) = M^{(i)}$ by induction on i.

(b) Suppose $I \trianglelefteq L$ and $(L/I)^{(n)} = 0$ for some n. Use (a) result for surjection $\pi : L \to L/I$ to get $\pi(L^{(n)}) = (L/I)^{(n)} = 0$, that is, $L^{(n)} \subseteq I = \ker(\pi)$. If $I^{(m)} = 0$ for some m, get $L^{(m+n)} = (L^{(n)})^{(m)} \subseteq I^{(m)} = 0$.

(c) By 3rd isom. theorem, $(I+J)/J \cong I/(I \cap J)$ so $I/(I \cap J)$ is solvable as a hom. image of solvable I. The isomorphism says $(I+J)/J$ is solvable, and J solvable, so by (b), $I+J$ is solvable. $\square$

Application:

Def. A maximal solvable ideal, S, of Lie algebra, L, is a solvable ideal not contained in any larger solvable ideal of L.

If $I \trianglelefteq L$ is any solv. ideal of L, then by last Thm (c), $S+I \trianglelefteq L$ is solvable so $S+I=S$ so $I \subseteq S$. Thus, S is the unique solvable ideal of L containing all solvable ideals of L.

Def. $Rad(L)$ is the unique maximal solvable ideal of L. L is semisimple when $Rad(L)=0$. $L/Rad(L)$ is semisimple.