<u>Def</u>. For Lie algebra L, the <u>descending central series</u> of L is the sequence of ideals $L^i \trianglelefteq L$ defined by $L^0 = L$, $L^1 = [L,L]$, $L^2 = [L,L^1]$, $L^3 = [L,L^2]$, ..., $L^i = [L, L^{i-1}]$. Say L is <u>nilpotent</u> when $L^n = 0$ for some n.

<u>Ex</u>: L abelian $\Rightarrow L^1 = 0 \Rightarrow L$ nilp.

L nilp $\Rightarrow L$ solv. since $L^{(i)} \subseteq L^i$ for all i.

<u>Note</u>: For $L = \mathfrak{t}(n,F)$ we have $L^{(1)} = L^1 = \mathfrak{n}(n,F)$ and $L^2 = [L, L^1] = L^1$ so $L^i = L^1$ for $i \geq 1$ so L is <u>not</u> nilp. but it is solv.

For $M = \mathfrak{n}(t,F)$, $M^1 = [M,M]$ has basis $\{E_{ij} \mid j-i \geq 2\}$, $M^2 = \langle E_{ij} \mid j-i \geq 3 \rangle$, ..., $M^k = \langle E_{ij} \mid j-i \geq k+1 \rangle$, so <u>M nilp</u>.

Th. Let L be a Lie algebra.

(a) If L is nilp. then so are its subalgebras and homomorphic images.

(b) If $L/Z(L)$ is nilp. then so is L.

(c) If $L \neq 0$ is nilp. then $Z(L) \neq 0$.

Pf. (a) If $K \leq L$ then $K^i \leq L^i$, $i \geq 0$.
If $\phi : L \to K$ is onto then $\phi(L^i) = K^i$.

(b) Suppose for some n that $L^n \leq Z(L)$. Then
$$L^{n+1} = [L, L^n] \subseteq [L, Z(L)] = 0.$$

(c) In the desc. central series, say
$$0 \neq L^{n-1} = [L, L^{n-2}] \text{ but } 0 = L^n = [L, L^{n-1}].$$
Then $L^{n-1} \leq Z(L)$ is non-zero. $\square$

$L^n = 0$ means $\forall x_1, x_2, \dots, x_n, y \in L$,
$0 = [x_1, [x_2, [\cdots [x_n, y] \cdots]]] = ad_{x_1} \circ ad_{x_2} \circ \cdots \circ ad_{x_n}(y)$,
so if all $x_i = x$, $(ad_x)^n = 0 \in End(L), \forall x \in L$.

<u>Def</u>. Say $x \in L$ is <u>ad-nilp</u> if $ad_x \in End(L)$ is a nilp. endom.

So L nilp Lie alg $\Rightarrow$ all $x \in L$ are ad-nilp.

<u>Th.</u> (Engel) If $\forall x \in L$, x is ad-nilp then L is a nilp. Lie algebra.

To prove Engel's Thm need some work first.

<u>Lemma</u>: Let $x \in gl(V)$ be a nilp. endom. Then $ad_x \in End(End(V))$ is also nilp.

Pf. Let $\lambda_x, \rho_x \in \mathrm{End}(\mathrm{End}(V))$ be left and right mult by x, that is, $\forall y \in \mathrm{End}(V)$,

$\lambda_x(y) = x \circ y$ and $\rho_x(y) = y \circ x$. Each of these is nilp since $(\lambda_x)^n(y) = x^n \circ y = 0$ if $x^n = 0$ and $(\rho_x)^n(y) = y \circ x^n = 0$. Certainly λ_x and ρ_x commute since $(\lambda_x \circ \rho_x)(y) = x(yx) = (xy)x$
$= (\rho_x \circ \lambda_x)(y)$ since $\mathrm{End}(V)$ is assoc.

In ring R (any) the sum or difference of commuting nilp. elts is nilp. from the binomial expansion:

$(f+g)^m = \sum_{i=0}^{m} \binom{m}{i} f^i g^{m-i}$. If $f^r = 0 = g^s$ then for $m = r+s-1$ we have

$f^i g^{m-i} = 0$ for $i \geq r$ and for $0 \leq i < r$
$m-i = r+s-1-i > s-1$ so $g^{m-i} = 0$, so all
terms in the sum are zero.

Apply this to $f = \lambda_x$ and $g = -\rho_x$ to get
$ad_x = \lambda_x - \rho_x$ is nilp. $\square$

<u>Note</u>: $I_V \in gl(V)$ is <u>not</u> nilp. since $I_V^n = I_V$
for all n, but it <u>is ad-nilp.</u> since
$ad_{I_V}(y) = I_V \circ y - y \circ I_V = 0_V^V$, $\forall y \in gl(V)$.
In terms of matrices, $I_n \in gl(n,F)$ is
not nilp, but is ad-nilp. with $ad_{I_n} = 0$
$\in End(gl(n,F))$. $d(n,F)$ and $\eta(n,F)$ are <u>both</u> nilp.

Th. Let $L \leq gl(V)$, $\dim(V) < \infty$, $V \neq 0$, L65
and $\forall x \in L$, x is nilp. endom. of V. Then
$\exists\, 0 \neq v \in V$ s.t. $L \cdot v = 0$.

If $\dim(L) = 0$ this
Pf. By induction on $\dim(L)$. If $\dim(L)=0$ this
is trivial. If $\dim(L) = 1$, $L = \langle x \rangle$ has a basis
x, a nilp. endom. of V. So x has e.value 0
from elem. lin. algebra and the $0 \neq v \in V$ we
seek is an e.vector for x with e.value 0.

Suppose $K \lneq L$ is any subalg. By last lemma,
$ad|_K : K \to gl(L)$ s.t. $\forall y \in K$, $ad_y : L \to L$ is
nilp. Since $ad_y(K) \subseteq K$, there is an induced
lin. map $\overline{ad_y} : L/K \to L/K$ by $\overline{ad_y}(x+K) = [y,x]+K$
and it is also nilp.

Since $\dim(K) < \dim(L)$, the inductive hypothesis of the theorm applies to $K \leq \mathfrak{gl}(L/K)$ and says $\exists\ x+K \neq K$ in L/K killed by all $\mathrm{ad}_y \in \mathfrak{gl}(L/K)$, that is, $\forall y \in K$, $[y,x] \in K$ but $x \notin K$. So $x \in N_L(K)$ and $K \subsetneqq N_L(K)$.

Choose $K \subsetneqq L$ to be a <u>maximal</u> proper subalg. of L, so discussion above gives $N_L(K) = L$, that is, $K \trianglelefteq L$, $K \neq L$, is an <u>ideal</u> of L.

Suppose $\dim(L/K) > 1$, then any 1-dim'l subspace of L/K is a Lie subalg of Lie alg. L/K. [redacted] $\pi : L \to L/K$ surj. Lie alg. hom. such that the pre-image under π of any subalg. of L/K is a subalg. $M \leq L$ s.t. $K \leq M$ and $\pi(M) = M/K$. If $\dim(M/K) = 1$ then $\dim(M) = 1 + \dim(K)$ but $\dim(L) - \dim(K) > 1$ so $\dim(L) > 1 + \dim(K) = \dim(M) > \dim(K)$ makes M a subalg., $K \lneq M \lneq L$, impossible for K max. proper subalg. of L.

Thus, $\dim(L/K) = 1$ $(\operatorname{codim}_L(K) = 1)$ and $L = K + \mathbb{F}z$ for any $z \in L \setminus K$.

By induction, $W = \{v \in V \mid K \cdot v = 0\} \neq 0$ since the inductive hypotheses apply to $K \leq \mathfrak{gl}(V)$.

Since $K \trianglelefteq L$ we have $\forall x \in L, \forall y \in K, \forall w \in W$, $[x,y] \in K$ and
$[x,y]\cdot w = (xy - yx)\cdot w$ so
$$y\cdot(x\cdot w) = x\cdot\underbrace{(y\cdot w)}_{=0} - \underbrace{[x,y]\cdot w}_{=0} = 0.$$
This means $L\cdot W \subseteq W$, W is "L-stable".
Choose $z \in L - K$ so that nilp. endom. z acting on W has an e.vector $0 \neq v \in W$ with e.value 0 for z, so $z\cdot v = 0$. Then
$$L\cdot v = (K + \mathbb{F}z)\cdot v = K\cdot v + \mathbb{F}z\cdot v = 0. \quad \square$$

We can now complete a proof of Engel's Thm using these results.

Pf. of Engel's Th. We may assume $L \neq 0$ is a Lie alg. whose elements are all ad-nilp. The Lie alg. $ad(L) = \{ad_x \mid x \in L\} \leq gl(L)$ (assuming $\dim(L) < \infty$) satisfies the assumptions of the last theorem. Then $\exists\, 0 \neq X \in L$ s.t. $[L, X] = 0$, so $X \in Z(L) \neq 0$. $L/Z(L)$ has smaller dimension than L, and as a hom. image of L consists of ad-nilp. elts. By induction on $\dim(L)$, Engel says $L/Z(L)$ is nilp so by Theorem on p. 61, part (b), L is nilp. $\square$

__Def.__ For vector space V with $\dim(V) = n < \infty$ a __flag__ in V is a chain of subspaces $0 = V_0 \leq V_1 \leq V_2 \leq \cdots \leq V_n = V$ with $\dim(V_i) = i$. Say $X \in \mathrm{End}(V)$ __stabilizes__ this flag when $X \cdot V_i \subseteq V_i$ for $0 \leq i \leq n$.

__Cor.__ With the hypotheses of the Theorem on page 65, $\exists$ flag $\{V_i\}$ in V __stable__ under L with $X \cdot V_i \subseteq V_{i-1}$ for all i. This means there is a basis of V, $S = \{v_1, \dots, v_n\}$ s.t. $S_i = \{v_1, \dots, v_i\}$ is a basis of V_i, $1 \leq i \leq n$, and $\forall x \in L$, the matrix representing X w.r.t. S is strictly upper triangular, in $\mathfrak{n}(n, F)$.

Pf. By Thm $\exists\, 0 \neq v \in V$, $L \cdot v = 0$. let $V_1 = Fv$ and let $W = V/V_1$. L acts on W by nilp endom.'s, so we can apply inductive argument on $\dim(V)$ to get a flag in W stabilized by L, $0 = W_0 < W_1 < \cdots < W_{n-1} = W$, with $X \cdot W_i \subseteq W_{i-1}$. $\pi : V \longrightarrow V/V_1 = W$ gives pre-images $V_1 = \pi^{-1}(W_0) < \pi^{-1}(W_1) = V_2 < \cdots < \pi^{-1}(W) = V$ which provide the desired flag in V. $\square$ [71]

Lemma (for later use): Let L be nilp, $K \trianglelefteq L$. If $K \neq 0$ then $K \cap Z(L) \neq 0$ so $Z(L) \neq 0$.

Pf. $\forall x \in L$, $ad_x : K \to K$ so $ad_x \in gl(K)$ are nilp. so $\exists\, 0 \neq y \in K$ s.t. $[L, y] = 0$ so $y \in K \cap Z(L)$. $\square$

Homework: Page 14, #4, 6