Assume $\mathrm{char}(F)=0$ and F is algebraically closed

Th. Let $L \leq \mathfrak{gl}(V)$ be soluble, $\dim(V)<\infty$, $V\neq 0$. Then $\exists\, 0\neq v\in V$ s.t. v is an e-vector for all $x\in L$, with e.value depending on x.

Pf. By induction on $\dim(L)$. Base case $\dim(L)=0$ is trivial. If $\dim(L)=1$, just take a basis element $x\in L$, as an End(V) its char. poly. has at least one root in alg. closed field F, so get at least one e.vector for x in V.

Outline of proof: step 1: Find $K \trianglelefteq L$ with $\dim(L/K)=1$; 2: Use ind. to find common e.vect. for K; 3 Show L stabilizes space of all common K e.vectors; 4 In that space, find an e.vector for some $z \in L$ s.t. $L = K + Fz$.

Step ①: L solv., $\dim(L) \geq 1$ so $[L,L] \trianglelefteq L$. $L/[L,L]$ is abelian so any subspace of it is an ideal. Take a subspace of codim 1, its inverse image under $\pi : L \to L/[L,L]$ is an ideal $K \trianglelefteq L$ s.t. $K \supsetneq [L,L]$ and $\dim(L/K) = 1$
so $\dim(L) = 1 + \dim(K)$.

Step ②: $K \subsetneq L \leq gl(V)$ is solv, so by induction $\exists\, 0 \neq v \in V$, common e-vector for K. If $K = 0$ then $\dim(L) = 1$ and we already discussed it.
$\forall x \in K$, $x \cdot v = \lambda(x) v$ for some linear map $\lambda : K \to F$. Fix that λ and define subspace

$W = \{ w \in V \mid x \cdot w = \lambda(x) w, \ \forall x \in K \} \neq 0$

since $v \in W$.

Step ③: Show $L \cdot W \subseteq W$. This is a long
argument completed below. Assuming it, do
Step ④: Write $L = K + Fz$ for some $z \notin K$.
F alg. closed so $z \in \mathrm{End}(V)$ has an e-vector
$v_0 \in W$ since $z \cdot W \subseteq W$ allows us to restrict
the action, get $z \in \mathrm{End}(W)$. Then v_0 is a
common e.vector for all of L since $L = K + Fz$.
Can extend $\lambda : L \to F$ s.t. $x \cdot v_0 = \lambda(x) v_0, \ \forall x \in L$.

Rest of the proof consists of doing Step ③
Show $L \cdot W \subseteq W$.

Let $w \in W$, $x \in L$, then $x \cdot w \in W$ iff $\forall y \in K$, $y \cdot (x \cdot w) = \lambda(y)\, x \cdot w$. But we know that

$$
\begin{aligned}
y \cdot (x \cdot w) &= x \cdot (y \cdot w) - [x,y] \cdot w \\
&= x \cdot (\lambda(x) w) - \lambda([x,y])\, w \\
&= \lambda(x)\, x \cdot w - \lambda([x,y])\, w
\end{aligned}
$$

So we must show that $\lambda([x,y]) = 0$.

Suppose $w \in W$ and $x \in L$ are fixed for the rest of this discussion. Let $n > 0$ be minimal s.t. $\{w, x \cdot w, \dots, x^n \cdot w\}$ is dependent. Let $W_0 = 0$, $W_i = \langle w, x \cdot w, \dots x^{i-1} w \rangle$ so $\dim(W_i) = i$ for $0 \le i \le n$ but $W_{n+1} = W_{n+2} = \dots = W_n$. Also, $x \cdot W_i \in W_{i+1}$ so $x \cdot W_n \subseteq W_{n+1} = W_n$.

How does any $y \in K$ act on each W_i?

we have: $y\cdot w = \lambda(y) w \in W_1$ so $y\cdot W_1 \subseteq W_1$

$$y\cdot(x\cdot w) = x\cdot(y\cdot w) - [x,y]\cdot w$$
$$= \lambda(y)\, x\cdot w - \lambda([x,y]) w \in W_2 \text{ so } y\cdot W_2 \subseteq W_2$$

$$y\cdot(x^2\cdot w) = y\cdot(x\cdot(x\cdot w)) = x\cdot(y\cdot(x\cdot w)) - [x,y]\cdot(x\cdot w)$$
$$= x\cdot\big(\lambda(y)\, x\cdot w - \lambda([x,y]) w\big) - [x,y]\cdot(x\cdot w)$$
$$= \underbrace{\lambda(y)\, x^2\cdot w}_{\in W_3} - \underbrace{\lambda([x,y])\, x\cdot w - [x,y]\cdot(x\cdot w)}_{\in W_2}$$

so $y\cdot W_3 \subseteq W_3$. Prove $y\cdot W_i \subseteq W_i$ by ind. on i.

$$y\cdot(x^{i-1}\cdot w) = y\cdot(x\cdot(x^{i-2}\cdot w)) = x\cdot(\underbrace{y\cdot(x^{i-2}\cdot w)}_{\in W_{i-1}}) - \underbrace{[x,y]\cdot(x^{i-2}\cdot w)}_{\in W_{i-1}}$$

$\in W_i$.

Look closer at matrix representing action of

$y \in K$ on W_n w.r.t. basis $\{w, x\cdot w, \dots, x^{n-1}\cdot w\}$.

<u>Claim</u>: That matrix is upper triangular with diagonal entries all equal to $\lambda(y)$.

<u>Pf</u>. This comes from formula

(*) $\quad y\cdot(x^i\cdot w) \equiv \lambda(y)\, x^i\cdot w \pmod{W_i}$ for $i \geq 0$.

That formula is proved by induction on i.

$i=0$: $y\cdot w = \lambda(y) w$ (recall $W_0 = 0$).

Assume true for $i-1$:

$$y\cdot(x^{i-1}\cdot w) = \lambda(y)(x^{i-1}\cdot w) + w' \text{ for } w' \in W_{i-1}$$

Then $y\cdot(x^i\cdot w) = y\cdot(x\cdot(x^{i-1}\cdot w)) = x\cdot(y\cdot(x^{i-1}\cdot w)) - [x,y]\cdot(x^{i-1}\cdot w)$

$$= x\cdot(\lambda(y)\, x^{i-1}\cdot w + w') - [x,y]\cdot(x^{i-1}\cdot w)$$

$$= \lambda(y)\, x^i\cdot w + x\cdot w' - [x,y]\cdot(x^{i-1}\cdot w) \quad \text{and}$$

$x \cdot w' \in W_i$ and $[x,y] \in K$ so $[x,y]\cdot(x^{i-1}\cdot w) \in W_i$.

Compute trace of endomorphism y on W_n, get

$$\mathrm{Tr}\begin{bmatrix} \lambda(y) & & * \\ & \lambda(y) & \\ 0 & & \ddots \;\lambda(y) \end{bmatrix}_{n\times n} = n\lambda(y).$$

Good for any elt. of K, so good for $[x,y] \in K$ (fixed $x \in L$, $y \in K$). Since x and y stabilize W_n and their commutator $[x,y] : W_n \to W_n$ and $\mathrm{Tr}([x,y]) = 0$ is true for any comm. of ops. on W_n, so $n\lambda([x,y]) = 0$. $\mathrm{char}(F) = 0 \Rightarrow$ $\lambda([x,y]) = 0$, completes the proof. $\square$

<u>Cor. A (Lie's Th.)</u>: Let $L \leq \mathfrak{gl}(V)$ be solv. for $\dim(V) = n < \infty$. Then L stabilizes some flag in V, so matrices representing elements of L w.r.t. an appropriate basis of V are all upper triangular, in $\mathfrak{t}(n, F)$.

<u>Pf</u>. By ind. on $n = \dim(V)$. Base case $n = 1$ is clear. For $\dim(V) = n > 1$, last Theorem gives $0 \neq v \in V$ s.t. $\forall x \in L$, $x \cdot v = \lambda(x) v$ for a linear map $\lambda : L \to F$. Let $V_1 = Fv$ and $W = V/V_1$ so $\dim(W) = \dim(V) - 1$. The action of each $x \in L$ induces an action on W, so have $L \leq \mathfrak{gl}(W)$. Inductive hypothesis applies to W, gives flag in W, L-stable. $\pi : V \longrightarrow V/V_1 = W$ as before

allows pre-images of W flag,
$0 = W_0 < W_1 < \cdots < W_{n-1} = W$ to give
$V_1 = \pi^{-1}(W_0) < V_2 = \pi^{-1}(W_1) < \cdots < V_n = \pi^{-1}(W_{n-1}) = V$
which is a flag in V if we add $V_0 = 0$ on left.

<u>Cor. B</u>: Let L be solv. Then $\exists$ chain of ideals of L; $0 = L_0 \leq L_1 \leq \cdots \leq L_n = L$ s.t. $\dim(L_i) = i$.

<u>Pf</u>. For any rep'n $\phi : L \to gl(V)$ with $\dim(V) < \infty$
$\phi(L)$ (hom image of L) is solv in $gl(V)$ so stabilizes some flag in V by Cor. A. For $ad : L \to gl(L)$, a flag of subspaces of L stable under ad_x is a flag of <u>ideals</u> of L with dim's as described above. $\square$

Cor. C: Let L be solv. Then $x \in [L,L]$ implies $ad_x : L \to L$ is nilp. so $[L,L]$ is nilp.

Pf. Let $0 = L_0 < L_1 < \cdots < L_n = L$ be a flag of ideals of L as in Cor. B. Let $\{x_1, \ldots, x_i\}$ be a basis of each L_i, $1 \le i \le n$, so w.r.t. this basis $\{x_1, \ldots, x_n\}$ of L, the matrices representing each ad_x are in $t(n,F)$, upper triangular. So matrices of commutators $[ad_x, ad_y] = ad_{[x,y]}$ are strictly upper triangular, in $\eta(n,F)$. These are all nilp. on L, and certainly $ad_{[x,y]} : [L,L] \to [L,L]$ are all nilp, so $[L,L]$ is nilp. by Engel's Th. $\square$