Jordan-Chevalley Decomposition

Assume F is alg. closed, but char(F) is arbitrary. Jordan block form for $x \in End(V)$, $\dim(V) < \infty$, is a block diagonal matrix representing x w.r.t. some basis, where the blocks look like

$$J(a,m) = \begin{bmatrix} a & 1 & & 0 \\ & a & 1 & \\ & & \ddots & 1 \\ 0 & & & a \end{bmatrix} \in F_m^m \text{ for } a \in F$$

$$= aI_m + \sum_{i=1}^{m-1} E_{i(i+1)}$$

$$= aI_m + N(m)$$

so $N(m)$ is nilpotent and $N(1) = [0] \in F_1^1$ and $N(m)^m = 0_m^m$.

Note: $[aI_m, N(m)] = 0_m^m$

If $J_x = \begin{bmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_r \end{bmatrix}$ is the Jordan block form matrix rep'ing x, so each $B_i = J(a_i, m_i)$ for some $a_i \in F$, $1 \le m_i \in \mathbb{Z}$, then

$\dim(V) = n = m_1 + m_2 + \cdots + m_r$, $a_1, \ldots, a_r$ need not be distinct, any power of J_X is also the Jordan block form matrix:

$$J_X^k = \begin{bmatrix} B_1^k & & & 0 \\ & B_2^k & & \\ & & \ddots & \\ 0 & & & B_r^k \end{bmatrix}$$ and $J_X = S_X + N_X$ where

$$S_X = \begin{bmatrix} a_1 I_{m_1} & & & 0 \\ & a_2 I_{m_2} & & \\ & & \ddots & \\ 0 & & & a_r I_{m_r} \end{bmatrix}$$ is a diagonal matrix and

$$N_X = \begin{bmatrix} N(m_1) & & & 0 \\ & N(m_2) & & \\ & & \ddots & \\ 0 & & & N(m_r) \end{bmatrix}$$ is a nilp. matrix which commutes with S_X.

Furthermore, the char. poly. of X is
$$\det(tI_n - J_X) = \prod_{i=1}^{r} (t-a_i)^{m_i}$$
but the minimal poly. of X is more subtle. There can be Jordan blocks with the same diagonal entry, $J(a_1,m_1)$ and $J(a_2,m_2)$ could have $a_1 = a_2 = a$, and the sizes can be arranged in non-increasing order $m_1 \geq m_2$. To simplify notation, fix $a \in \{a_1, \dots, a_r\}$ and let all the Jordan blocks in J_X with diagonal entry a be listed with sizes as $J(a, m_{a1}), J(a, m_{a2}), \dots, J(a, m_{an_a})$, that is, $m_{a1} \geq m_{a2} \geq \dots \geq m_{an_a} \geq 1$ gives n_a J-blocks with e-value a in the sizes indicated with m_{a1} the largest size.

The usual Jordan form matrix J_X is made with its sub-blocks organized into groups, one for each distinct e-value, say $b_1, \ldots, b_t$, and for each one, $b = b_k$, $1 \le k \le t$, we have

$$J(b) = \begin{bmatrix} J(b, m_{b1}) & & & \\ & J(b, m_{b2}) & & \\ & & \ddots & \\ & & & J(b, m_{bn_b}) \end{bmatrix}$$

and $J_X = \begin{bmatrix} J(b_1) & & & \\ & J(b_2) & & \\ & & \ddots & \\ & & & J(b_t) \end{bmatrix}$ grouping the t distinct e-values into blocks, each of which has the sub-block structure shown above. Letting $J(b_j) \in F^{k_j}_{k_j}$ for size

$k_j = m_{b_j1} + m_{b_j2} + \cdots + m_{b_jn_{b_j}}$ (so many subscripts!)

the char. poly. of X can be written as

$$\prod_{j=1}^{t}(t-b_j)^{k_j}$$

which assumes $b_1, \ldots, b_t$ are distinct, with apologies for using t in two ways! But now we can find

$$m_X(t) = \prod_{j=1}^{t}(t-b_j)^{m_j}$$

the min. poly. of X

where the power $m_j = m_{b_j}$, is the size of the largest Jordan block for b_j in $J(b_j)$.

<u>Def</u>. k_j is called the <u>algebraic multiplicity</u> of the eigenvalue b_j for operator $X \in End(V)$.

$g_j = \dim(X_{b_j})$ = <u>geometric mult.</u> of b_j for X, where $X_{b_j} = \{v \in V \mid X \cdot v = b_j v\}$ is the b_j e-space of X.

Note. Using our notation for $J(b_j)$, the first column of each J-block $J(b_j, m_{b_j}\, i)$ for $1 \le i \le n_{b_j}$ corresponds to a basis e-vector with e-value b_j. So $g_j = n_{b_j}$ is the number of J-blocks in $J(b_j)$.

Def. For $\dim(V) < \infty$, say $X \in End(V)$ is **semisimple** when min. poly. of X is $m_X(t) = \prod_{j=1}^{t} (t - b_j)$ is a product of distinct linear factors, so all $m_j = 1$, with no repeated roots. For F alg. closed this means X is diagonalizable.

So X is diag-able when each $J(b_j)$ is diagonal, so $J(b_j) = b_j I_{k_j}$ and $n_{b_j} = k_j = g_j$ (alg. mult. = geom. mult. for each j).

<u>Lemma</u>: If $X, Y \in End(V)$ are commuting semisimple operators on V, they can be simultaneously diagonalized, that is, V has a basis w.r.t. which both X and Y are rep'd by diagonal matrices.

<u>Pf.</u> There is a direct sum decomposition $V = \bigoplus_{i=1}^{m} V_{\lambda_i}$ into e-spaces for Y. For any $v \in V_{\lambda_i}$, $Y \cdot v = \lambda_i v$ so $Y \cdot (X \cdot v) = X \cdot (Y \cdot v) = \lambda_i (X \cdot v)$

so $X(V_{\lambda_i}) \subseteq V_{\lambda_i}$

Since X is diag-able on V and stabilizes each e-space for y, X restricts to a diag-able map on each V_{λ_i}. The min. poly. of the restriction divides the $m_X(t)$, so it is also a product of distinct linear factors.

There is an e-basis of each V_{λ_i} for X, and all such vectors are e-vectors for y, so they are simultaneous e-vectors for X and y. ∎

__Cor.__ If $X, y \in End(V)$ are commuting s.s. op's on V then $X \pm y$ are also semisimple.

__Note__: If $X \in End(V)$ is s.s. and for $W \leq V$ we have $X \cdot W \subseteq W$ then $X|_W : W \to W$ is s.s.

<u>Prop.</u> Let $x \in End(V)$ for $\dim(V) < \infty$.

(a) $\exists! x_s, x_n \in End(V)$ s.t. $x = x_s + x_n$ for x_s semisimple, x_n nilp. and $[x_s, x_n] = 0$.

(b) $\exists\, p(T), q(T) \in F[T]$ s.t. $p(0) = 0 = q(0)$ and $x_s = p(x)$, $x_n = q(x)$. $\forall y \in End(V)$ s.t. $[x, y] = 0$ we have $[x_s, y] = 0 = [x_n, y]$.

(c) If $A \subset B \subset V$ are subspaces of V and $x \cdot B \subseteq A$ then $x_s \cdot B \subseteq A$ and $x_n \cdot B \subseteq A$.

<u>Note</u>: $x = x_s + x_n$ is called the Jordan-Chevalley decomposition of x into its semisimple and nilpotent parts.

Pf. For F alg. closed the char. poly of x is $\prod_{i=1}^{k}(T-a_i)^{m_i}$ for distinct e-values $a_1,\dots,a_k$ with alg. multiplicities $m_1,\dots,m_k$ (Humphreys' notation). Let $V_i = \ker(x-a_iI_V)^{m_i}$ so that $V=\bigoplus_{i=1}^{k} V_i$ and $x\cdot V_i \subseteq V_i$ for $1\le i\le k$.

The restriction $x|_{V_i}: V_i \to V_i$ has char. poly. $(T-a_i)^{m_i}$. The polynomials $(T-a_1)^{m_1},\dots,(T-a_k)^{m_k}$ are pairwise relatively prime. If all $a_i \neq 0$ then these are also relatively prime to T. If one of the e-values is 0 then there is a T^{m_i} in the list for some i.

The Chinese Remainder Theorem (CRT) is a basic ring theory result which applies to ring $F[T]$ and gives the existence of a poly. $p(T)$ satisfying the congruences:

$$p(T) \equiv a_i \pmod{(T-a_i)^{m_i}} \text{ for } 1 \le i \le k \text{ and}$$
$$p(T) \equiv 0 \pmod{T},$$

which is implied by one above if some $a_i = 0$, given by CRT if all $a_i \neq 0$ since T is rel. prime to each $(T-a_i)^{m_i}$.

Let $q(T) = T - p(T)$, so $p(0) = 0 = q(0)$, and let $x_s = p(x)$, $x_n = q(x)$. These are both polys in x so $[x_s, x_n] = 0$ and $\forall y \in End(V)$ s.t. $[x, y] = 0$ we have $[x_s, y] = [p(x), y] = 0$ and $[x_n, y] = [q(x), y] = 0$.

Any subspace of V stabilized by X is also stabilized by $p(X)$ and $q(X)$, for example, each generalized e-space V_i. Using congruence

$$p(T) \equiv a_i \pmod{(T-a_i)^{m_i}}$$ we see that

$$p(X) - a_i I_V = X_s - a_i I_V = (X - a_i I_V)^{m_i} \text{ ... is } 0 \text{ on } V_i$$

So $X_s|_{V_i} = a_i I_{V_i}$ for each $1 \le i \le k$ makes X_s semisimple on V. We defined

$$X_n = q(X) = X - p(X) = X - X_s$$

so on V_i we have $X_n^{m_i} = (X - a_i I_{V_i})^{m_i} = 0$

and then X_n is nilp on all of V, killed by the power $\mathrm{Max}(m_1, \ldots, m_k)$.

(c) is true because for $A \subset B \subset V$ with $X \cdot B \subseteq A$ we have $X^i \cdot B \subseteq X^{i-1} \cdot (X \cdot B) \subseteq X^{i-1} \cdot A \subseteq X^{i-1} \cdot B \subseteq A$ by induction on $i \geq 1$.

Note that $X^0 \cdot B = B$ need not be in A, but $X_s = p(X)$ and $X_n = q(X)$ only involve powers of X^i for $i \geq 1$ since $p(T)$ and $q(T)$ have 0 constant terms.

To finish the proof we only need to show the uniqueness of X_s and X_n in part (a).

Suppose there is another decomposition $X = s + n$ where s is s.s., n is nilp., $[s, n] = 0$.

Then $X_s - s = n - X_n$ and $[s, X] = 0 = [n, X]$ so by (b), $[s, X_s] = 0 = [s, X_n]$, $[n, X_s] = 0 = [n, X_n]$.

Thus, $X_s - S$ is a difference of two commuting s.s. endo's of V, so it is s.s., and $n - X_n$ is a diff. of two comm. nilp. endo's of V, so it is nilp.

The only operator in End(V) which is both s.s. and nilp. is the zero operator, 0^V_V.

So $s = X_s$ and $n = X_n$ gives uniqueness. $\square$

<u>Applications:</u> For $\dim(V) < \infty$ let $L = gl(V)$ and its rep'n $ad : L \longrightarrow End(L)$.

We have seen that $x \in gl(V)$ nilp. $\Rightarrow ad_x$ nilp.

<u>Lemma</u>: If $x \in gl(V)$ is s.s. then ad_x is s.s.

<u>Pf</u>. Let $S = \{v_1, \dots, v_n\}$ be a basis of V such that