Thus, $X_s - s$ is a difference of two commuting s.s. endo's of V, so it is s.s., and $n - X_n$ is a diff. of two comm. nilp. endo's of V, so it is nilp.

The only operator in $End(V)$ which is both s.s. and nilp. is the zero operator, 0_V^V.

So $s = X_s$ and $n = X_n$ gives uniqueness. $\square$

<u>Applications:</u> For $\dim(V) < \infty$ let $L = \mathfrak{gl}(V)$ and its rep'n $ad : L \longrightarrow End(L)$.

We have seen that $x \in \mathfrak{gl}(V)$ nilp. $\Rightarrow ad_x$ nilp.

<u>Lemma</u>: If $x \in \mathfrak{gl}(V)$ is s.s. then ad_x is s.s.

<u>Pf</u>. Let $S = \{v_1, \ldots, v_n\}$ be a basis of V such that

$X \cdot v_i = a_i v_i$ so the matrix rep'ing X w.r.t. S is $\mathrm{diag}(a_1, \ldots, a_n)$. The "std. basis" of $gl(V)$ associated with S is the set $\{\mathcal{E}_{ij} \in \mathrm{End}(V) \mid 1 \le i, j \le n\}$ where $\mathcal{E}_{ij}(v_k) = \delta_{jk} v_i$ for $1 \le k \le n$. The $n \times n$ matrix representing $\mathcal{E}_{ij}$ w.r.t. S is $E_{ij} \in F_n^n$ which has 1 in its $(i,j)^{th}$ entry, 0's everywhere else.

To compute $ad_X(\mathcal{E}_{ij})$ we apply it to any v_k:

$$(X \circ \mathcal{E}_{ij} - \mathcal{E}_{ij} \circ X) v_k = \delta_{jk} a_i v_i - a_k \delta_{jk} v_i$$

$$= (\delta_{jk} a_i - \delta_{jk} a_j) v_i = (a_i - a_j) \delta_{jk} v_i = (a_i - a_j) \mathcal{E}_{ij} v_k$$

so $ad_X(\mathcal{E}_{ij}) = (a_i - a_j) \mathcal{E}_{ij}$ shows ad_X is s.s. $\square$

Lemma A: Let $x \in End(V)$ for $\dim(V) < \infty$,
and let $x = x_s + x_n$ be its Jordan decomposition.
Then $ad_x = ad_{x_s} + ad_{x_n}$ is the J-decomp.
of $ad_x \in End(End(V))$.

Pf. We know ad_{x_s} is s.s. and ad_{x_n} is nilp.
and $[ad_{x_s}, ad_{x_n}] = ad_{[x_s, x_n]} = ad_0 = 0$.
The uniqueness of J-decomp. from Prop. (a)
gives the result. $\square$

Lemma B. Let $(\mathcal{a}, *)$ be any fin. dim'l F-alg.
Then $Der(\mathcal{a}) = \{\delta \in End(\mathcal{a}) \mid \delta(x*y) = \delta(x)*y + x*\delta(y), \forall x,y \in \mathcal{a}\}$
and $\forall \delta \in Der(\mathcal{a})$, the s.s.
and nilp. parts of δ, δ_s and δ_n, are each in $Der(\mathcal{a})$.

<u>Pf.</u> Let $\delta \in Der(\mathfrak{A})$, $\sigma = \delta_s$, $\nu = \delta_n \in End(\mathfrak{A})$ so $\delta = \sigma + \nu$. We will show $\sigma \in Der(\mathfrak{A})$, and that will imply $\nu = \delta - \sigma \in Der(\mathfrak{A})$.

For $a \in F$ let $\mathfrak{A}_a = \{x \in \mathfrak{A} \mid (\delta - aI)^k x = 0$ for some k (dep'ing on x)$\}$.

Then $\mathfrak{A} = \bigoplus \mathfrak{A}_a$ where the sum is over those $a \in F$ s.t. a is an e.value for δ (or for σ) and $\sigma : \mathfrak{A}_a \to \mathfrak{A}_a$ so that the restriction $\sigma|_{\mathfrak{A}_a} = a I_{\mathfrak{A}_a}$ is scalar mult by $a \in F$.

For any $a, b \in F$, $\forall\, x, y \in \mathfrak{A}$ we have

$$(*) \quad (\delta - (a+b)I)^n (x * y) = \sum_{i=0}^{n} \binom{n}{i} \left((\delta - aI)^{n-i} x\right) * \left((\delta - bI)^i y\right)$$

for $0 \le n \in \mathbb{Z}$, by induction on n.

The $n=0$ case is trivial: $x*y = x*y$.

For $n=1$: $(\delta-(a+b)I)(x*y) = \delta(x*y)-(a+b)(x*y)$

$= \delta(x)*y + x*\delta(y) - a(x*y) - b(x*y)$

$= \big((\delta - aI)(x)\big)*y + x*\big((\delta-bI)y\big)$.

Inductive Step: Exercise.

(Did you use assoc. of $*$ in $\mathfrak{a}$ at any point?)

Then $(*)$ implies $\mathfrak{a}_a * \mathfrak{a}_b \subseteq \mathfrak{a}_{a+b}$. (Check!)

If $x \in \mathfrak{a}_a$, $y \in \mathfrak{a}_b$ then $\sigma(x*y) = (a+b)(x*y)$ since $x*y \in \mathfrak{a}_{a+b}$ where σ acts as scalar $a+b$.

But then $\sigma(x*y) = \sigma(x)*y + x*\sigma(y) = (ax)*y + x*(by)$ and $\mathfrak{a} = \bigoplus \mathfrak{a}_a$ means $\sigma \in Der(\mathfrak{a})$. $\square$

Cartan's Criterion (for solvability of L)

Note: If $[L,L]$ is nilp. then L is solvable since $L^{(1)} = [L,L] = L^1$ and $L^{(i)} \subseteq L^i$ for $i \geq 0$.

Also: $[L,L]$ is nilp. iff $\forall x \in [L,L]$, $ad_x : [L,L] \to [L,L]$ is nilp. by Engel's Th.

So we first develop a trace criterion for $x \in End(V)$ to be nilp.

Lemma: Let $A \subset B \subset gl(V)$ be subspaces with $\dim(V) < \infty$. Let $M = \{x \in gl(V) \mid [x, B] \subset A\}$. Suppose $x \in M$ satisfies $Tr(xy) = 0$, $\forall y \in M$. Then x is nilp.

Pf. Let $x = s+n$ be the Jordan decomposition of x, so $s = x_s$, $n = x_n$ in previous notation.
Let $\mathcal{S} = \{v_1, \dots, v_m\}$ be a basis of V s.t. matrix of s is $\mathrm{diag}(a_1, \dots, a_m)$, that is, $s \cdot v_i = a_i v_i$. Let
$E = \mathbb{Q}a_1 + \cdots + \mathbb{Q}a_m = \mathbb{Q}$-span of $\{a_1, \dots, a_m\} \subseteq F$
considered as a vector space over $\mathbb{Q} \subseteq F$ since
$\mathrm{char}(F) = 0$. To show x is nilp. we want to
show $s = 0$, that is, $E = 0$. $\dim_{\mathbb{Q}}(E) < \infty$ so
$\dim(E) = \dim(E^*)$, dual space, so we want to
show that any lin. map $f : E \to \mathbb{Q}$ is zero.
$\forall f \in E^*$, let $y \in \mathfrak{gl}(V)$ be defined by $y \cdot v_i = f(a_i) v_i$
for $1 \le i \le m$, so the matrix rep'ing y w.r.t. $\mathcal{S}$
is $\mathrm{diag}(f(a_1), \dots, f(a_m))$. As before, let $\{E_{ij} \in \mathfrak{gl}(V) \mid 1 \le i, j \le m\}$ be the

basis of $gl(V)$ s.t. $E_{ij}(v_k) = \delta_{jk} v_i$ and so
$ad_s(E_{ij}) = (a_i - a_j)E_{ij}$ and $ad_y(E_{ij}) = (f(a_i) - f(a_j))E_{ij}$
for $1 \le i, j \le m$ since s and y are semisimple.
Let $r(T) \in F[T]$ be a poly with $r(0) = 0$ s.t.
$r(a_i - a_j) = f(a_i) - f(a_j)$ for all $1 \le i, j \le m$.
Such a poly. is found by Lagrange interpolation.
If $a_i - a_j = a_k - a_l$ then lin. of f gives the consistent value $f(a_i - a_j) = f(a_k - a_l)$. Then, evaluation of poly. $r(T)$ at $T = ad_s$ yields the endomorphism $r(ad_s) \in End(gl(V))$ such that

$$r(ad_s)(E_{ij}) = r(a_i - a_j)E_{ij} = (f(a_i) - f(a_j))E_{ij}$$
$$= ad_y(E_{ij}) \text{ so } r(ad_s) = ad_y.$$