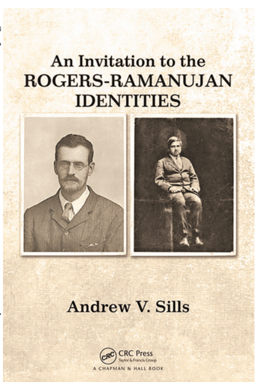
BOOK REVIEW

An Invitation to the Rogers–Ramanujan Identities

Reviewed by Krishnaswami Alladi



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An Invitation to the Rogers–Ramanujan Identities
By Andrew V. Sills
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In the entire theory of partitions and q-hypergeometric series (q-series, for short), the two *Rogers–Ramanujan identities* are unmatched in simplicity of form, elegance, and depth:

$$R_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-4})(1-q^{5m-1})}, \tag{1}$$

and

$$R_2(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{5m-3})(1-q^{5m-2})} \tag{2}$$

for $|q| < 1$. The partition interpretation of the identities is given below.

The identities have a fascinating history. The Indian mathematical genius Srinivasa Ramanujan (1887–1920) noticed that the ratio $R_1(q)/R_2(q)$ admits a continued fraction expansion, which in view of (1) and (2) enjoys a product representation. More precisely, the Ramanujan continued fraction identity is

$$\rho(q) := \frac{R_1(q)}{R_2(q)} = 1 + \frac{q}{1+\frac{q^2}{1+\frac{q^3}{1+\cdots}}} = \prod_{m=1}^{\infty} \frac{(1-q^{5m-3})(1-q^{5m-2})}{(1-q^{5m-4})(1-q^{5m-1})} \quad \text{for } |q|<1. \tag{3}$$

More importantly, Ramanujan realized that the ratio $q^{1/5}/\rho(q)$ satisfies transformation properties and used them to make the incredible evaluation

$$\frac{q^{-2\pi/\sqrt{5}}}{\rho(e^{-2\pi\sqrt{5}})} = \frac{\sqrt{5}}{1+(5^{3/4}(\frac{\sqrt{5}-1}{2})^{5/2}-1)^{1/5}} - \frac{\sqrt{5}+1}{2}. \tag{4}$$

He sent (4) along with dozens of startling results of great variety in his first of two letters in 1913 to G. H. Hardy (1877–1947) of Cambridge University (see [4, p. 8]). It was a result such as this that made Hardy realize that Ramanujan was a genius in the class of Euler or Jacobi for sheer manipulative ability.

Ramanujan's continued fraction plays a central role in the theory of modular forms and has links with the fundamental elliptic modular function $j(\tau)$. We provide a brief description of this now.

The modular group $\Gamma(1)$ is the set of 2×2 matrices with integer entries having determinant 1, where a matrix A is identified with its negative $-A$. This is because $\Gamma(1)$ is

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Figure 1. Srinivasa Ramanujan, FRS (1887–1920).

Figure 2. G. H. Hardy, FRS (1877–1947).

identified with the set of mappings

$$\phi(\tau) = \frac{a\tau + b}{c\tau + d}$$

where $\mathrm{Im}(\tau) > 0$ and a, b, c, d are integers satisfying $ad - bc = 1$. A modular function on $\Gamma(1)$ is a meromorphic function f on the upper half-plane that satisfies

$$f(\phi(\tau)) = f(\tau)$$

for all such mappings ϕ and has, at worst, a pole at $i\infty$. The elliptic modular function $j(\tau)$ is a modular function on $\Gamma(1)$ and is called a *Hauptmodul* for $\Gamma(1)$ because every modular function on $\Gamma(1)$ can be expressed as a rational function of j. One can generalize this discussion by considering subgroups of the modular group, and one important class of subgroups comprises the principal congruence subgroups $\Gamma(N)$, namely, the members of $\Gamma(1)$ that are congruent to the identity modulo N. It turns out that $q^{1/5}/\rho(q)$ is a Hauptmodul for the congruence subgroup $\Gamma(5)$, where $|q| < 1$ is represented as $q = e^{2\pi i\tau}$, with $\mathrm{Im}(\tau) > 0$.

Although (1) and (2) form the source for (3), when Ramanujan arrived in England in 1914 and Hardy asked for proofs of (1) and (2), Ramanujan could not provide them. Interestingly, in 1917, when Ramanujan was going through certain old issues of the *Journal of the London Mathematical Society*, he accidentally came across a trilogy of papers dating back to 1893–95 of the British mathematician L. J. Rogers (1862–1933), in which (1), (2), and (3), and many identities similar to (1) and (2) were proved [6]. Rogers was a brilliant mathematician, who in Hardy's own admission was a man of talents similar to Ramanujan (see [4, p. 91]), but his work was ignored by his British peers. Indeed, it was Ramanujan's discovery of Rogers's work that led to Rogers's belated recognition.

As this drama was playing out in England, Issai Schur (1875–1941) in Germany, cut off from England due to World War I, had simultaneously and independently discovered and proved the Rogers–Ramanujan identities and stated their partition (combinatorial) version as well. Actually Hardy's colleague Major MacMahon (1854–1929) had stated the partition version of (1) and (2), but in 1915 since he was not aware of Rogers's proof, he stated them as unproved conjectures in his monumental book on combinatorial analysis [5]. MacMahon was a wizard in computing, and he even assisted in numerically verifying the celebrated Hardy–Ramanujan asymptotic formula for partitions, but before becoming a mathematician he had a successful career in the British military, and that was how he was known as Major MacMahon even in the mathematical world.

Neither Rogers nor Ramanujan stated the combinatorial (partition) interpretation of (1) and (2), and so the following partition theorem is independently due to MacMahon and Schur:

Partition version of the Rogers–Ramanujan identities. *For $i = 1, 2$, the number of partitions of an integer into parts that differ by ≥ 2 and least part $\geq i$ is equal to the number of partitions of that integer into parts $\equiv \pm i \pmod 5$.*

Since Schur emphasized the partition version of (1) and (2) as above, he was able to discover the next partition theorem with parts in congruence classes $\pm 1 \pmod 6$ connecting to partitions with parts differing by ≥ 3, but he needed the extra condition that there should be no

Figure 3. Leonard James Rogers, FRS (1862–1933).

consecutive multiples of 3. And this is where the real story of Rogers–Ramanujan-type identities begins, which leads us to this book of Sills.

A Rogers–Ramanujan-type (R-R type) identity is an

$$\text{infinite} \quad q\text{-series} = \text{infinite} \quad q\text{-product}$$

identity, where the series is the generating function of partitions whose parts satisfy gap conditions, and the product is the generating function of partitions whose parts satisfy congruence conditions. The term R-R type derives from (1) and (2), which are the prototype. The very first partition theorem that is of this type is due to Euler, who founded the theory of partitions and q-series. Euler's theorem states that the number of partitions of an integer into distinct parts is equal to the number of partitions of that integer into odd parts; this can be analytically expressed in the form

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{2m-1})}.$$

Schur viewed the distinct parts condition as gap ≥ 1 between parts, and odd parts as those $\equiv \pm 1 \pmod 4$. From this viewpoint, (1) is the "next level" partition theorem (but much deeper!) because the gap condition is ≥ 2 and the congruence is $\pm 1 \pmod 5$. Schur's theorem itself deals with gaps ≥ 3, with the additional condition that there are no consecutive multiples of 3 connecting to the congruence condition for parts $\pm 1 \pmod 6$. As the modulus of the congruence increases, the gap conditions become more complex (see Andrews [1, Ch. 7] for a good discussion of R-R type identities).

In the 1940s, W. N. Bailey (1893–1961) found a mechanism [3], now called the *Bailey chain*, to generate R-R type identities. This was developed by his student Lucy Slater (1922–2008), who published a list [7] of more than 100 R-R type identities. The full potency of the Bailey chain was revealed by George Andrews, who in the 1960s and 1970s led the development of a systematic theory of R-R type identities both from a q-series and partition theoretic point of view (see [2, Ch. 3]). Andrews was also inspired by Basil Gordon's lovely and important generalization of the Rogers–Ramanujan partition theorem to all odd moduli (1961). Today, R-R type identities are playing a central role in (i) the theory of modular forms, (ii) conformal field theory and statistical mechanics, and (iii) Lie algebras (vertex operators).

When the residue classes can be paired off as conjugates (mod M) in the product form of an R-R type identity, such an identity has links with the theory of modular forms, because the conjugacy of the residue classes means that these products can be represented in terms of theta functions by the use of Jacobi's fundamental triple product identity:

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{m=1}^{\infty} (1-q^{2m})(1+zq^{2m-1})(1+z^{-1}q^{2m-1}).$$

(In the product above, if one replaces q by $q^{M/2}$ and chooses $z = q^{(M-2a)/2}$, for some integer $a \in [1, M/2]$, then one would be dealing with conjugate residue classes modulo M because of the presence of z and z^{-1}.) A beautiful example is the pair of Gollnitz–Gordon identities,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}(1+q)(1+q^3)\cdots(1+q^{2n-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{8m-7})(1-q^{8m-4})(1-q^{8m-1})}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(1+q)(1+q^3)\cdots(1+q^{2n-1})}{(1-q^2)(1-q^4)\cdots(1-q^{2n})} = \prod_{m=1}^{\infty} \frac{1}{(1-q^{8m-5})(1-q^{8m-4})(1-q^{8m-3})},$$

where the two products can be written as ratios of certain theta functions. These identities are to the modulus 8 what the Rogers–Ramanujan identities are to the modulus 5. L. J. Rogers found several such identities connected to the modulus 5 or its multiples, and many of his identities (together with (1) and (2)) turned out to play a central role in Rodney Baxter's solution (1979) of the hard-hexagon model in statistical mechanics (see Andrews [2, Chs. 1

and 8]). From a study of vertex operators in Lie algebra theory beginning with Lepowsky and Wilson, several new R-R type identities have been found in the last quarter century, spurred by the discovery of Capparelli's partition theorems (1992). There are also very deep and important R-R type identities where the residue classes (mod M) are not a set of conjugates. A fine example is the (Big) partition theorem of Göllnitz (1967). Such identities are not related to modular forms but are nevertheless very fundamental.

In the book, Sills simply refers to all R-R type identities as Rogers–Ramanujan identities. He starts with the foundations laid by Euler and walks you through the works of Ramanujan and Rogers and discusses all the topics I have mentioned above and much more. For instance, he includes a discussion of Rogers–Ramanujan identities in knot theory and their connections with the dilogarithm function and with the Hall–Littlewood polynomials.

The Hall–Littlewood polynomials in n-variables are certain symmetric functions indexed by partitions of n and also involving a parameter q. For certain generalizations of the well-known Andrews–Gordon extension of the Rogers–Ramanujan identities to all odd moduli, Sills discusses the connection between the series side of such identities and the Hall–Littlewood polynomials (work of M. Griffin, K. Ono, and O. Warnaar (2016)). Knot theory is the study of the topology of knots, and various invariants involving polynomials and groups have been used to gain a deeper understanding of them. In recent years q-series have arisen in knot theory, and Sills discusses a few q-series that arise in the study of certain simple fundamental knots. While there have been systematic discussions of many of the topics mentioned above in various books, monographs, and survey articles, Sills's book is the first comprehensive discussion of R-R type identities in all their forms, describing the state of the art. Since the subject is so vast, he does not provide proofs for most of the identities discussed, but he provides an interesting and illuminating historical context for each topic, gives good motivation, and describes the key ideas underlying the proofs. Thus the word "invitation" in the title is most appropriate, and the exposition will appeal to nonexperts. He also provides a substantial number of references that will lead both the student and the expert to some of the most important sources in the field. Especially valuable to the researcher is the list of 200 R-R type identities (extending the Slater list substantially), classifying them according to moduli in increasing size. Of historical value is the correspondence of Bailey with Slater and Freeman Dyson that Sills has presented in the appendices. The world-renowned physicist Dyson started out working on partitions and q-series as an undergraduate at Cambridge University, inspired by the work of Ramanujan. G. H. Hardy assigned the young Dyson, twenty years old then in 1943, to referee a certain paper of Bailey on his mechanism to generate Rogers–Ramanujan type identities. Dyson had some penetrating comments on Bailey's work, and so even though he was the referee, Hardy informed Bailey as to who the referee was in sending the comments. A correspondence between Bailey and Dyson ensued that lasted until 1946.

Finally, I should say that, aptly, George Andrews, as a leader in the field and as the former PhD advisor of Andrew Sills, has written a fine foreword to the book.

In summary, this is a comprehensive and easily readable treatment of R-R type identities of appeal to both the expert and the potential entrant to the field. It will be a fine addition to both libraries and your personal book collection.

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