JOURNAL OF ALGEBRA 156, 433–453 (1993)

Representations of Hyperbolic Kac–Moody Algebras

ALEX J. FEINGOLD*

Department of Mathematical Sciences, The State University of New York, Binghamton, New York 13902-6000

IGOR B. FRENKEL†

Department of Mathematics, Yale University, P.O. Box 2155, Yale Station, New Haven, Connecticut 06520

AND

JOHN F. X. RIES‡

Department of Mathematical Sciences, The State University of New York, Binghamton, New York 13902-6000

Communicated by Nathan Jacobson

Received April 13, 1991

1. INTRODUCTION

The theory of Kac–Moody Lie algebras, which originated in the works of Kac [K1], Moody [Mo], and Kantor [Kan], has been developing in two rather unrelated directions. In one direction, the affine Kac–Moody algebras, or central extensions of loop algebras, and their representations were found to be related to numerous mathematical and physical theories (see [Fr2, G] for reviews). This continues to be an extensive and promising field of research. In another direction, the general theory of Kac–Moody algebras and their representations has developed more slowly into a somewhat isolated field with few applications. For the most part, efforts have been directed toward generalizing various results from the theory of finite dimensional simple Lie algebras and the corresponding

* Work partially supported by National Science Foundation Grants DMS-8420318 and DMS-8703156, and by the Institute for Advanced Study in Princeton, New Jersey.

† Work partially supported by the Sloan Foundation, by National Science Foundation Grant DMS-8602091, and by the Institute for Advanced Study in Princeton, New Jersey.

‡ Current Address: Mathematics Department, East Carolina University, Greenville, North Carolina 27858.

0021-8693/93 $5.00

groups. This work has often required sophisticated generalizations of the classical techniques. (See, e.g., [Mat] for an excellent example of this program.)

The fundamental difficulties in the general theory or Kac–Moody algebras appear because we lack an understanding of their internal structure and, in particular, their geometrical nature. Very few facts are known which are really intrinsic to the non-affine Kac–Moody algebras. Even those few which were obtained indicate an enormous richness of structure and a possibly deep connection of the non-affine algebras with number theory: e.g., the theory of quadratic forms, modular forms, adeles, etc. In the theory of non-affine algebras the first problem encountered is the difficult one of determining root and weight multiplicities. In [FF] we began to study a certain rank 3 hyperbolic Kac–Moody algebra, $\mathcal{F}$, which we believe can be thought of as the simplest generic example of non-affine Kac–Moody algebras. We identified the root system and weight lattice of $\mathcal{F}$ with certain integral binary quadratic forms, and the Weyl group as the extended modular group, $PGL(2, \mathbb{Z})$, acting naturally on quadratic forms. This immediately implies that the root and weight multiplicities are integer valued functions of the classes, and as such should have a number theoretic meaning. The hyperbolic algebra $\mathcal{F}$ has an affine subalgebra $\mathcal{F}_0$ of type $A_1^{(1)}$, and the central element of that subalgebra provides a $\mathbb{Z}$-grading $\mathcal{F} = \sum_{n \in \mathbb{Z}} \mathcal{F}_n$. We refer to $\mathcal{F}_n$ as the nth *level* of $\mathcal{F}$. Using the representation theory of the affine subalgebra, in [FF] we found all the root multiplicities on levels $n = 0, \pm 1, \pm 2$. Our work was further developed in [Fr1, B1, KMW, Ka1, Ka2]. In [B1] Borcherds introduced a class of generalized Kac–Moody algebras, and in some special cases he determined the root multiplicities. This allowed him to deduce from the "denominator formula" some remarkable new identities for modular functions, some of them related to Monstrous Moonshine [B2].

The method of constructing $\mathcal{F}$ given in [FF] was influenced by the work of Kantor [Kan], and has a straightforward generalization to the construction of other graded algebras; for example, the Lorentzian Kac–Moody algebras. In this paper we show how our methods can be applied to the construction of modules for these algebras, and to the determination of the weight multiplicities on the first few levels. In Section 2 we explicitly construct certain representations of graded Lie algebras. This parallels the construction in [FF] of the graded algebra itself from its local part. Each module is explicitly described as a quotient of an induced representation of $\mathbf{g} = \bigoplus_{n \in \mathbb{Z}} \mathbf{g}_n$ from a representation of $\mathbf{g}^+ = \bigoplus_{n \geqslant 0} \mathbf{g}_n$ which is a trivial extension of a representation of $\mathbf{g}_0$. We apply this construction to hyperbolic and Lorentzian Kac–Moody algebras in Section 3. If the subalgebra $\mathbf{g}_0$ is an affine Kac–Moody algebra, then the Dynkin diagram of $\mathbf{g}$ is obtained from the diagram of $\mathbf{g}_0$ by adding one point, connected by one line to one point of the affine diagram. In this case, our construction of representations gives all irreducible highest weight modules for $\mathbf{g}$, and we obtain an explicit description of the first two or three levels of the module in terms of $\mathbf{g}_0$-modules. Finally, we apply these results to the determination of level 2 weight multiplicities in fundamental modules for the rank 3 hyperbolic algebra $\mathscr{F}$.

We end the Introduction with several remarks on prospects for further work. Since our construction of representations of graded Lie algebras is quite general, it can be used in a great variety of settings. For example, it applies to the construction of highest weight representations of finite dimensional simple Lie algebras, representations of graded Lie algebras of Cartan type, and representations of the generalized Kac–Moody Lie algebras of Borcherds. A more detailed analysis of our construction of modules for the Lorentzian Kac–Moody algebras, combined with the techniques of Kang, should allow the extension of our explicit weight multiplicity formulas to a few more levels. As already mentioned, an important unsolved problem in Kac–Moody theory is the determination of *all* weight multiplicities (even in just one module), preferably in a Weyl group invariant fashion. Our results can be used to find and test conjectures about the answer to this problem. Thus, our present work should be considered as one of the very first steps toward the understanding of the intrinsic structure of the highest weight modules for non-affine Kac–Moody algebras.

2. Constructions of Graded Algebras and Representations

Let

$$\mathbf{g}_{\mathrm{loc}} = \mathbf{g}_{-1} \oplus \mathbf{g}_0 \oplus \mathbf{g}_1 \tag{2.1}$$

be a vector space over $\mathbb{C}$. Suppose there is an antisymmetric bilinear product (called "bracket")

$$[\ ,\ ]: \mathbf{g}_i \times \mathbf{g}_j \to \mathbf{g}_{i+j} \tag{2.2}$$

defined when $i, j, i+j \in \{0, \pm 1\}$, and such that the Jacobi identity is satisfied when the brackets are defined. In particular, this says that $\mathbf{g}_0$ is a Lie algebra and $\mathbf{g}_{-1}$ and $\mathbf{g}_1$ are $\mathbf{g}_0$-modules. Define a $\mathbb{Z}$-graded Lie algebra

$$\bar{G} = \sum_{n \in \mathbb{Z}} \bar{G}_n, \tag{2.3}$$

where $\bar{G}_n = \mathbf{g}_n$ for $-1 \leqslant n \leqslant 1$ and where

$$\bar{G}^+ = \sum_{n \geqslant 1} \bar{G}_n \qquad \text{and} \qquad \bar{G}^- = \sum_{n \geqslant 1} \bar{G}_{-n} \tag{2.4}$$

and the free Lie algebras generated by $\bar{G}_1$ and $\bar{G}_{-1}$, respectively. So for $n \geqslant 1$, $\bar{G}_n$ (respectively, $\bar{G}_{-n}$) is spanned by all formal brackets of n vectors from $\bar{G}_1$ (respectively, $\bar{G}_{-1}$). There is a canonical extension of the bracket (2.2) from $\mathbf{g}_{\text{loc}}$ to all of $\bar{G}$ such that the Jacobi identity is satisfied on $\bar{G}$. The Lie algebra $\bar{G}$ satisfies

$$[\bar{G}_i, \bar{G}_j] \subset \bar{G}_{i+j}, \qquad i, j \in \mathbb{Z}, \tag{2.5}$$

and contains two canonical graded ideals [FF, Section 4]

$$J^{\pm} = \sum_{n > 1} J_{\pm n}, \tag{2.6}$$

where

$$J_{\pm n} = \{x \in \bar{G}_{\pm n} \mid [y_1, [y_2, \dots [y_{n-1}, x] \dots]] = 0 \text{ for all } y_i \in \bar{G}_{\mp 1}\}. \tag{2.7}$$

Then

$$J = J^+ \oplus J^- \tag{2.8}$$

is a natural ideal in $\bar{G}$, and we call the $\mathbb{Z}$-graded quotient

$$\mathbf{g} = \bar{G}/J \tag{2.9}$$

the Lie algebra associated with $\mathbf{g}_{\text{loc}}$. In many important applications we will see that $\mathbf{g}$ is a simple Lie algebra.

For any vector space V we denote by $T(V)$ the tensor algebra over V,

$$T(V) = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots, \tag{2.10}$$

and by $S(V)$ the symmetric algebra over V,

$$S(V) = T(V)/I, \tag{2.11}$$

where I is the ideal of $T(V)$ generated by $\{v_1 \otimes v_2 - v_2 \otimes v_1 \mid v_1, v_2 \in V\}$. Denote by V^* the dual space of linear functionals on V.

We now construct a representation of $\mathbf{g}$. Let U be any $\mathbf{g}_0$-module and let $V = \mathbf{g}_{-1}$. Define the graded $\mathbf{g}_0$-module

$$H = T(V) \otimes U = \sum_{n \geqslant 0} H_{-n}, \tag{2.12}$$

where $H_{-n} = V \otimes \cdots \otimes V \otimes U$ with n tensor factors of V for $n \geqslant 0$. We can extend the action of $\mathbf{g}_0$ on H to an $\mathbf{g}_{\rm loc}$ as follows. For $v_1, ..., v_n \in V$, $v_{n+1} \in U$, let $h = v_1 \otimes \cdots \otimes v_n \otimes v_{n+1} \in H_{-n}$ and define

$$x \cdot h = x \otimes h \qquad \text{if} \quad x \in \mathbf{g}_{-1} = V, \tag{2.13}$$

$$x \cdot h = \sum_i v_1 \otimes \ldots (x \cdot v_i) \ldots v_{n+1} \qquad \text{if} \quad x \in \mathbf{g}_0, \tag{2.14}$$

$$x \cdot h = \sum_{i<j} v_1 \otimes \ldots \hat{v}_i \ldots [x, v_i] \cdot v_j \ldots v_{n+1} \qquad \text{if} \quad x \in \mathbf{g}_1, \tag{2.15}$$

where $\hat{v}_i$ means v_i was removed and the summations run from 1 to $n+1$. This action of $\mathbf{g}_{\rm loc}$ uniquely determines an action of $\bar{G}$ on H (for example, $[x, y] \cdot h = x \cdot (y \cdot h) - y \cdot (x \cdot h)$ for $h \in H$, $x, y \in \bar{G}^{\pm}$) such that

$$\bar{G}_m : H_{-n} \to H_{m-n}, \qquad m \in \mathbb{Z}, \quad n \geqslant 0. \tag{2.16}$$

PROPOSITION 2.1. *H is a $\bar{G}$-module.*

Proof. H is a $\bar{G}$-module if for all $x, y \in \bar{G}$, $h \in H$,

$$[x, y] \cdot h = x \cdot (y \cdot h) - y \cdot (x \cdot h). \tag{2.17}$$

For $x, y \in \bar{G}^{\pm}$, the right side of (2.17) is the definition of the left side. It is sufficient to verify (2.17) for $x, y \in \mathbf{g}_{\rm loc}$. Since (2.14) is the usual definition of H as a $\mathbf{g}_0$-module, (2.17) is valid for $x, y \in \mathbf{g}_0$. Let $h = v_1 \otimes \ldots \otimes v_{n+1} \in H_{-n}$.

If $x \in \mathbf{g}_0$, $y \in \mathbf{g}_{-1}$, then $x \cdot y = [x, y] \in \mathbf{g}_{\rm loc}$, so $x \cdot (y \cdot h) = x \cdot (y \otimes h) = (x \cdot y) \otimes h + y \otimes (x \cdot h) = [x, y] \cdot h + y \cdot (x \cdot h)$.

If $x \in \mathbf{g}_1$ and $y \in \mathbf{g}_{-1}$ then $x \cdot (y \cdot h) = x \cdot (y \otimes h) = \hat{y} \otimes [x, y] \cdot h + y \otimes (x \cdot h) = [x, y] \cdot h + y \cdot (x \cdot h)$.

Let $x \in \mathbf{g}_0$, $y \in \mathbf{g}_1$ and let

$$\sum{}' = \sum_{\substack{i<j \\ k \neq i, j}} v_1 \otimes \ldots \hat{v}_i \ldots (x \cdot v_k) \ldots [y, v_i] \cdot v_j \ldots v_{n+1}.$$

Then we have

$$x \cdot (y \cdot h) = \sum{}' + \sum_{i<j} v_1 \otimes \ldots \hat{v}_i \ldots (x \cdot ([y, v_i] \cdot v_j)) \ldots v_{n+1},$$

$$y \cdot (x \cdot h) = \sum{}' + \sum_{i<j} v_1 \otimes \ldots \hat{v}_i \ldots ([y, x \cdot v_i] \cdot v_j) \ldots v_{n+1}$$

$$+ \sum_{i<j} v_1 \otimes \ldots \hat{v}_i \ldots ([y, v_i] \cdot (x \cdot v_j)) \ldots v_{n+1},$$

and

$$[x, y] \cdot h = \sum_{i<j} v_1 \otimes \dots \hat{v}_i \dots ([[x, y], v_i] \cdot v_j) \dots v_{n+1}.$$

The Jacobi identity in $\mathbf{g}_{\mathrm{loc}}$ and the cases already established give

$$[[x, y], v_i] \cdot v_j = x \cdot ([y, v_i] \cdot v_j) - [y, [x, v_i]] \cdot v_j - [y, v_i] \cdot (x \cdot v_j) \tag{2.18}$$

which finishes the proof. ∎

We can explicitly describe the action of $\bar{G}$ on H as determined by the action of $\mathbf{g}_{\mathrm{loc}}$. For $x \in \bar{G}^-$ the action is obvious from (2.13). For $x \in \bar{G}^+$ we have the following result.

DEFINITION. For $x_1, \dots, x_i$ in any Lie algebra let

$$[x_1, x_2, \dots, x_i] = [\dots [[x_1, x_2], x_3] \dots, x_i].$$

PROPOSITION 2.2. *Let* $x \in \bar{G}_k$, $k \geqslant 1$, *and* $h = v_1 \otimes \cdots \otimes v_{n+1} \in H_{-n}$. *Then we have*

$$x \cdot h = \sum_{i_1 < \cdots < i_{k+1}} v_1 \otimes \dots \hat{v}_{i_1} \dots \hat{v}_{i_k} \dots \otimes [x, v_{i_1}, v_{i_2}, \dots, v_{i_k}] \cdot v_{i_{k+1}} \otimes \cdots \otimes v_{n+1}.$$

Proof. The statement is true for $k = 1$ by (2.15). Assuming the statement is true for $x \in \bar{G}_{k-1}$, it is sufficient to show it is true for $[x, y] \in \bar{G}_k$, where $y \in \bar{G}_1$. We have

$$[x, y] \cdot h = x \cdot (y \cdot h) - y \cdot (x \cdot h), \tag{2.19}$$

where

$$x \cdot (y \cdot h) = x \cdot \left(\sum_{i<j} v_1 \otimes \dots \hat{v}_i \dots [y, v_i] \cdot v_j \dots v_{n+1} \right) \tag{2.20}$$

and

$$y \cdot (x \cdot h) = y \cdot \Bigg(\sum_{i_1 < \cdots < i_k} v_1 \otimes \dots \hat{v}_{i_1} \dots \hat{v}_{i_{k-1}} \\ \dots [x, v_{i_1}, \dots, v_{i_{k-1}}] \cdot v_{i_k} \dots v_{n+1} \Bigg). \tag{2.21}$$

When we expand (2.20) (respectively, (2.21)) using the inductive assumption (respectively, using (2.15)) there are three types of terms: those where $[y, v_i] \cdot v_j$ (respectively, $[x, v_{i_1}, \dots, v_{i_{k-1}}] \cdot v_{i_k}$) is neither removed nor acted on, those where it is not removed but is acted on, and those where it is removed to act on another factor. The first types of terms in (2.20) and in (2.21) are the same, so are cancelled in (2.19). The remaining terms give

$$
\begin{aligned}
[x, y] \cdot h = & \sum_{i_1 < \cdots < i_{k+1}} \sum_{1 \leqslant n \leqslant k} v_1 \otimes \ldots \hat{v}_{i_1} \ldots \hat{v}_{i_k} \ldots [x, v_{i_1}, \ldots \hat{v}_{i_n} \ldots, v_{i_k}] \\
& \cdot ([y, v_{i_n}] \cdot v_{i_{k+1}}) \ldots v_{n+1} \\
& + \sum_{i_1 < \ldots < i_{k+1}} \sum_{1 \leqslant n < r \leqslant k} v_1 \otimes \ldots \hat{v}_{i_1} \ldots \hat{v}_{i_k} \ldots \\
& \ldots [x, v_{i_1}, \ldots \hat{v}_{i_n} \ldots ([y, v_{i_n}] \cdot v_{i_r}) \ldots, v_{i_k}] \cdot v_{i_{k+1}} \ldots v_{n+1} \\
& - \sum_{i_1 < \ldots < i_{k+1}} \sum_{1 \leqslant n \leqslant k} v_1 \otimes \ldots \hat{v}_{i_1} \ldots \hat{v}_{i_k} \ldots [y, v_{i_n}] \\
& \cdot ([x, v_{i_1}, \ldots \hat{v}_{i_n} \ldots, v_{i_k}] \cdot v_{i_{k+1}}) \ldots v_{n+1} \\
& - \sum_{i_1 < \ldots < i_{k+1}} v_1 \otimes \ldots \hat{v}_{i_1} \ldots \hat{v}_{i_k} \ldots [y, [x, v_{i_1}, \ldots, v_{i_{k-1}}] \cdot v_{i_k}] \\
& \cdot v_{i_{k+1}} \ldots v_{n+1}.
\end{aligned} \tag{2.22}
$$

The proposition then follows from the identity

$$
\begin{aligned}
& \sum_{1 \leqslant n \leqslant k} [[x, v_{i_1}, \ldots \hat{v}_{i_n} \ldots, v_{i_k}], [y, v_{i_n}]] \\
& \quad + \sum_{1 \leqslant n < r \leqslant k} [x, v_{i_1}, \ldots \hat{v}_{i_n} \ldots, [[y, v_{i_n}], v_{i_r}], \ldots, v_{i_k}] \\
& \quad - [y, [x, v_{i_1}, \ldots v_{i_{k-1}}, v_{i_k}]] \\
& = [[x, y], v_{i_1}, \ldots, v_{i_k}]
\end{aligned} \tag{2.23}
$$

which is valid in any Lie algebra. The proof of (2.23) by induction goes as follows. The case $k = 1$,

$$[x, [y, v]] - [y, [x, v]] = [[x, y], v],$$

is just the Jacobi identity. The left side of (2.23) can be written as

$$
\begin{aligned}
& \sum_{1 \leqslant n < k} [[x, v_{i_1}, \ldots \hat{v}_{i_n} \ldots, v_{i_k}], [y, v_{i_n}]] + [[x, v_{i_1}, \ldots, v_{i_{k-1}}], [y, v_{i_k}]] \\
& \quad + \sum_{1 \leqslant n < r < k} [x, v_{i_1}, \ldots \hat{v}_{i_n} \ldots, [[y, v_{i_n}], v_{i_r}], \ldots, v_{i_k}] \\
& \quad + \sum_{1 \leqslant n < k} [x, v_{i_1}, \ldots \hat{v}_{i_n} \ldots, [[y, v_{i_n}], v_{i_k}]] \\
& \quad - [[y, [x, v_{i_1}, \ldots, v_{i_{k-1}}]], v_{i_k}] - [[x, v_{i_1}, \ldots, v_{i_{k-1}}], [y, v_{i_k}]],
\end{aligned} \tag{2.24}
$$

whose second and last terms cancel. Combining the first and fourth terms of (2.24) using the Jacobi identity, the result is the left side of the inductively assumed identity bracketed with v_{i_k}, which equals $[[x, y], v_{i_1}, ..., v_{i_k}]$. ∎

COROLLARY 2.3. *J^+ acts trivially on H.*

Proof. This follows from (2.7), (2.16), and Proposition 2.2. ∎

Thus, we obtain a representation on H of the quotient Lie algebra $\bar{G}/J^+$. It is clear from (2.13) that $\bar{G}^-$, and J^- in particular, acts faithfully on H. In order to construct a $\mathbf{g}$-module we must factor H by a $\bar{G}$-submodule K such that $J^-: H \to K$. First note that $\bar{G}_{-2}$ can be naturally identified with the wedge product $V \wedge V$ (antisymmetric tensors in $V \otimes V$).

PROPOSITION 2.4. *Let W be a $\mathbf{g}_0$-submodule of $V \wedge V$. Let $K = K(W)$ be the subspace of H spanned by all subspaces $V \otimes ... \otimes W \otimes ... \otimes V \otimes U$ having one W factor. Then K is a $\bar{G}$-submodule of H if and only if $W \subseteq J_{-2}$.*

Proof. K will be a $\bar{G}$-submodule of H if and only if

$$\mathbf{g}_i: K \to K \qquad \text{for} \quad i = -1, 0, 1. \tag{2.25}$$

It is clear that (2.25) is true for $i = 0, -1$. Let

$$w = \sum_k (w_1^k \otimes w_2^k - w_2^k \otimes w_1^k) \in W \tag{2.26}$$

and

$$h_K = v_1 \otimes ... v_i \otimes w \otimes v_{i+1} ... \otimes v_{n+1} \in K, \tag{2.27}$$

where $v_1, ..., v_n \in V$, $v_{n+1} \in U$, and let $x \in \mathbf{g}_1$. Then from (2.15) it is easy to see that

$$\begin{aligned} x \cdot h_K = p_K + v_1 \otimes ... v_i \otimes \sum_k ([x, w_1^k] \cdot w_2^k - [x, w_2^k] \cdot w_1^k) \\ \otimes v_{i+1} ... v_{n+1}, \end{aligned} \tag{2.28}$$

where $p_K \in K$. The Jacobi identity in $\bar{G}$ gives

$$[[x, w_1^k], w_2^k] - [[x, w_2^k], w_1^k] = [x, [w_1^k, w_2^k]] \tag{2.29}$$

so

$$x \cdot h_K = p_K + v_1 \otimes ... v_i \otimes \left[x, \sum_k [w_1^k, w_2^k] \right] \otimes v_{i+1} ... v_{n+1}. \tag{2.30}$$

Under the identification of $\bar{G}_{-2}$ with $V \wedge V$,

$$\left[x, \sum_k [w_1^k, w_2^k]\right] = [x, w] \in \bar{G}_{-1} = V, \tag{2.31}$$

so K will be a $\bar{G}$-submodule of H if and only if $[x, w] = 0$ for all $x \in \mathbf{g}_1$, $w \in W$, that is, $W \subseteq J_{-2}$. ∎

We consider first the special case where $W = J_{-2} = \bar{G}_{-2}$, so $J_{-n} = \bar{G}_{-n}$ for all $n \geqslant 2$ and $K = I \otimes U$ is the kernel of the symmetrization map

$$\mathfrak{s} : H \to H_{\text{sym}}, \tag{2.32}$$

where (recall (2.11))

$$H_{\text{sym}} = S(V) \otimes U \tag{2.33}$$

and

$$\mathfrak{s}(v_1 \otimes \ldots \otimes v_n \otimes u) = (v_1 \ldots v_n) \otimes u. \tag{2.34}$$

To show that $J^- : H \to K$ it is enough to show that $J_{-2} : H \to K$ because $J_{-n-1} = [\bar{G}_{-1}, J_{-n}]$ and K is a $\bar{G}$-module. From (2.13) it is clear that $\bar{G}_{-2} : H \to K$. So in this case $\mathbf{g} = \bar{G}/J$ is represented on the quotient space $H/K \cong H_{\text{sym}}$. One can give explicit formulas for the action of $\mathbf{g}$ on H_{sym}. In particular, from (2.15) one has

$$\begin{aligned} x \cdot (v_1 \ldots v_n \otimes u) = \tfrac{1}{2} \sum_{i \neq j} (v_1 \ldots \hat{v}_i \ldots [x, v_i] \cdot v_j \ldots v_n) \otimes u \\ + \sum_i (v_1 \ldots \hat{v}_i \ldots v_n) \otimes [x, v_i] \cdot u \end{aligned} \tag{2.35}$$

for $x \in \mathbf{g}_1$.

The space $S(V)$ is well known in physics as the symmetric Fock space, and it occurs as a representation space for the Heisenberg algebra. It is a special case of our construction if we set

$$\begin{aligned} &\mathbf{g}_{-1} = V, \qquad \mathbf{g}_1 = V^*, \qquad \mathbf{g}_0 \in \mathbb{C}e, \\ &[v, v^*] = \langle v, v^* \rangle e \qquad \text{for} \quad v \in V, v^* \in V^*, \end{aligned} \tag{2.36}$$

where $\langle v, v^* \rangle$ is the natural pairing, and choose U to be a non-trivial one dimensional $\mathbf{g}_0$-module. Another example involves Hermitian symmetric pairs. In that case we have

$$\mathbf{g} = \mathbf{g}_{-1} \oplus \mathbf{g}_0 \oplus \mathbf{g}_1 \tag{2.37}$$

and our construction is equivalent to the holomorphic induced representation [H]. A third example is the algebra of vector fields on an n dimensional manifold (see, e.g., [K1]), where

$$\mathbf{g} = \sum_{m \geqslant -1} \mathbf{g}_m. \tag{2.38}$$

If $J_{-2} \neq \bar{G}_{-2}$ one should not expect to get a representation of $\mathbf{g}$ in a symmetric Fock space. Instead of $S(V)$ we can only obtain a "weakly commutative" algebra. We only consider the case, which is important for applications, where J^- is generated as a $\bar{G}$ ideal by J_{-2}.

PROPOSITION 2.5. *Let*

$$J^- = [\bar{G}, J_{-2}], \tag{2.39}$$

and let $W = J_{-2}$ *define* $K = \sum_{n \geqslant 2} K_{-n}$ *as in Proposition* 2.4, *where* $K_{-n} \subseteq H_{-n}$ *is the sum of the subspaces* $V^{\otimes i} \otimes W \otimes V^{\otimes n-i-2} \otimes U$ *for* $0 \leqslant i \leqslant n-2$. *Then we have* $J_{-2}\colon H \to K$, $J^-\colon H \to K$, $\mathbf{g}$ *is represented on* H/K, *and*

$$\begin{aligned} H/K = U \oplus (V \otimes U) \oplus (V \otimes V \otimes U)/(J_{-2} \otimes U) \\ \oplus \ldots \oplus H_{-n}/K_{-n} \oplus \ldots . \end{aligned} \tag{2.40}$$

3. CONSTRUCTIONS OF KAC–MOODY ALGEBRAS AND REPRESENTATIONS

Let $S \subset \mathbb{Z}$ be a non-empty finite index set, let $A = [A_{ij}]_{i,j \in S}$ be an indecomposable symmetrizable generalized Cartan matrix, and let $\mathbf{g} = \mathbf{g}(A)$ be the associated Kac–Moody Lie algebra. Then $\mathbf{g}$ has generators e_i, f_i, h_i, $i \in S$, with the relations

$$\begin{aligned} &[h_i, h_j] = 0, && [e_i, f_j] = \delta_{ij} h_i, && \\ &[h_i, e_j] = A_{ij} e_j, && [h_i, f_j] = -A_{ij} f_j && \text{for} \quad i, j \in S, \\ &(\operatorname{ad} e_i)^{1-A_{ij}} e_j = 0, && (\operatorname{ad} f_i)^{1-A_{ij}} f_j = 0 && \text{for} \quad i \neq j. \end{aligned} \tag{3.1}$$

If S' is a non-empty proper subset of the index set S, then the submatrix $A' = [A_{ij}]_{i,j \in S'}$ may be decomposable but still corresponds to a Kac–Moody algebra $\mathbf{g}(A')$ which is the direct sum of algebras associated with the indecomposable blocks of A'. We assume familiarity with the classification of the finite type Cartan matrices whose associated algebras are finite dimensional. They are the Cartan matrices which are positive definite. Another important class of Kac–Moody algebras is the class of affine algebras, those whose matrices A are degenerate positive semidefinite with all submatrices A' of finite type. If A is non-degenerate indefinite that each submatrix A' is decomposable into blocks of finite or affine type, then A is of hyperbolic type and $\mathbf{g}(A)$ is called a hyperbolic algebra. Although there are infinitely many rank 2 hyperbolic algebras $\mathbf{g}(A)$, $A = \left[\begin{smallmatrix} 2 & a \\ b & 2 \end{smallmatrix}\right]$ with $ab > 4$, there are only finitely many hyperbolic algebras with rank $\geqslant 3$, and the maximal rank is 10.

A Dynkin diagram corresponding to the Cartan matrix A has vertices labeled by the elements of S with edges between the ith and jth vertices iff $A_{ij} \neq 0$. If A is of finite type and $A_{ij} \neq 0$ then $1 \leqslant A_{ij}A_{ji} \leqslant 3$, and one has a single, double, or triple edge between the ith and jth vertices accordingly. If $A_{ij} \neq A_{ji}$ then a multiple edge is given an arrow pointing towards the ith vertex if $|A_{ij}| > |A_{ji}|$. If A is of affine type then $A_{ij}A_{ji} = 4$ is possible, so four edges (with or without an arrow) are used. There is a vertex (maybe more than one) in each affine diagram whose removal yields a connected diagram of finite type. We label such a point by $0 \in S$, and label the rest by $1 \leqslant i \leqslant l$. Dynkin diagrams for rank 2 hyperbolic algebras are given an edge labeled by a pair of negative integers (a, b) with $ab > 4$, but higher rank hyperbolics have $0 \leqslant A_{ij}A_{ji} \leqslant 4$ so the previous diagram scheme still works. In each hyperbolic diagram with more than two vertices there is a vertex whose removal yields a connected affine diagram, and we label such a vertex by $-1 \in S$. We can define a class of algebras, including the hyperbolics of rank at least three, for which representations can be constructed by the methods of Section 2. We call A *Lorentzian* if its Dynkin diagram is obtained by adjoining one vertex, labeled -1, connected by a single edge to one 0 point of an affine diagram.

For simplicity of exposition we suppose that $\mathbf{g} = \mathbf{g}(A)$, $A = [A_{ij}]_{-1 \leqslant i,j \leqslant l}$, is a Lorentzian algebra whose Dynkin diagram has no arrows (so $A_{ij} = A_{ji}$). Then the subalgebra $\mathbf{g}'' = \mathbf{g}(A'')$, $A'' = [A_{ij}]_{1 \leqslant i,j \leqslant l}$, generated by e_i, f_i, h_i, $1 \leqslant i \leqslant l$, is a finite-dimensional simple Lie algebra of type A_l $(l \geqslant 1)$, D_l $(l \geqslant 4)$, or E_l $(6 \leqslant l \leqslant 8)$. The subalgebra $\mathbf{g}' = \mathbf{g}(A')$, $A' = [A_{ij}]_{0 \leqslant i,j \leqslant l}$, generated by e_i, f_i, h_i, $0 \leqslant i \leqslant l$, is an affine algebra of type $A_l^{(1)}$ $(l \geqslant 1)$, $D_l^{(1)}$ $(l \geqslant 4)$, or $E_l^{(1)}$ $(6 \leqslant l \leqslant 8)$. We then say that the Lorentzian algebra $\mathbf{g}$ is of type $\hat{A}_l^{(1)}$ $(l \geqslant 1)$, $\hat{D}_l^{(1)}$ $(l \geqslant 4)$, or $\hat{E}_l^{(1)}$ $(6 \leqslant l \leqslant 8)$.

We can give another construction [FK] of $\mathbf{g}''$ from A'' as follows. Let Q be a rank l lattice with $\mathbb{Z}$-basis $\alpha_1, ..., \alpha_l$ and define a non-degenerate symmetric bilinear form on Q by $\langle \alpha_i, \alpha_j \rangle = A_{ij}$. Then Q is an even lattice (i.e., $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for $\alpha \in Q$) and we define

$$\Delta = \{\alpha \in Q \mid \langle \alpha, \alpha \rangle = 2\}. \tag{3.2}$$

Extend the form $\langle\ ,\ \rangle$ linearly to the vector space

$$\mathbf{h}^* = Q \otimes_{\mathbb{Z}} \mathbb{C} \tag{3.3}$$

and use it to identify $\mathbf{h}^*$ with its dual space

$$\mathbf{h} = \sum_{1 \leqslant i \leqslant l} \mathbb{C}h_i \tag{3.4}$$

by

$$\alpha_i(h_j) = A_{ij}. \tag{3.5}$$

There is a unique cohomology class

$$[\varepsilon] \in H^2(Q, \{\pm 1\}) \tag{3.6}$$

which can be represented by a bilinear 2-cocycle ε on Q satisfying

$$\varepsilon(\alpha, \beta)\, \varepsilon(\beta, \alpha)^{-1} = (-1)^{\langle \alpha, \beta \rangle}, \qquad \alpha, \beta \in Q. \tag{3.7}$$

Then $\mathbf{g}''$ is the vector space

$$\mathbf{g}'' = \mathbf{h} \oplus \sum_{\alpha \in \Delta} \mathbb{C}x_\alpha \tag{3.8}$$

with Lie algebra brackets

$$\begin{gathered} [h_i, h_j] = 0, \qquad [h, x_\alpha] = \alpha(h)\, x_\alpha, \\ [x_\alpha, x_\beta] = \begin{cases} 0 & \text{if } \langle \alpha, \beta \rangle \geqslant 0 \\ \varepsilon(\alpha, \beta)\, x_{\alpha+\beta} & \text{if } \langle \alpha, \beta \rangle = -1 \\ \varepsilon(\alpha, -\alpha)\alpha & \text{if } \langle \alpha, \beta \rangle = -2, \end{cases} \end{gathered} \tag{3.9}$$

for $h \in \mathbf{h}$, $\alpha, \beta \in \Delta$. From (3.2), for $\alpha, \beta \in \Delta$ we have $\alpha + \beta \in \Delta$ iff $\langle \alpha, \beta \rangle = -1$, and $\alpha + \beta = 0$ iff $\langle \alpha, \beta \rangle = -2$. We can extend the form $\langle\ ,\ \rangle$ to an invariant non-degenerate symmetric bilinear form on all of $\mathbf{g}''$ by setting

$$\begin{gathered} \langle h, x_\alpha \rangle = 0, \\ \langle x_\alpha, x_\beta \rangle = \begin{cases} 0 & \text{if } \alpha + \beta \neq 0 \\ \varepsilon(\alpha, -\alpha) & \text{if } \alpha + \beta = 0 \end{cases} \end{gathered} \tag{3.10}$$

for $h \in \mathbf{h}$, $\alpha, \beta \in \Delta$. We can choose ε such that

$$\varepsilon(\alpha, -\alpha) = -1 \qquad \text{for} \quad \alpha \in \Delta, \tag{3.11}$$

which simplifies (3.9) and (3.10). The identification of this construction with the one given by generators and relations is then

$$x_{\alpha_i} = e_i, \quad x_{-\alpha_i} = -f_i, \quad h_i = h_i, \qquad 1 \leqslant i \leqslant l. \tag{3.12}$$

With $\alpha_1, ..., \alpha_l \in \Delta$ distinguished as the simple roots of $\mathbf{g}''$, let $\alpha^+ \in \Delta$ denote the highest root. The fundamental weights $\omega_1, ..., \omega_l$ of $\mathbf{g}''$ are determined by $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$, $1 \leqslant i, j \leqslant l$.

We associate with $\mathbf{g}''$ an affine algebra

$$\hat{\mathbf{g}}'' = \mathbf{g}'' \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \tag{3.13}$$

with Lie brackets

$$\begin{gathered} [x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\delta_{m,-n}\langle x, y\rangle c, \\ [d, x \otimes t^m] = mx \otimes t^m, \qquad [c, x \otimes t^m] = 0, \qquad [c, d] = 0, \end{gathered} \tag{3.14}$$

for $x, y \in \mathbf{g}''$, $m, n \in \mathbb{Z}$. Let

$$x(m) = x \otimes t^m \tag{3.15}$$

and identify $\mathbf{g}'' \subset \hat{\mathbf{g}}''$ by $x = x(0)$. There is a unique extension of the form $\langle\ ,\ \rangle$ from $\mathbf{g}''$ to $\hat{\mathbf{g}}''$ given by

$$\begin{gathered} \langle x(m), y(n)\rangle = \langle x, y\rangle\, \delta_{m,-n}, \qquad \langle x(m), c\rangle = 0, \\ \langle x(m), d\rangle = 0, \qquad \langle c, c\rangle = 0, \qquad \langle d, d\rangle = 0, \qquad \langle c, d\rangle = 1, \end{gathered} \tag{3.16}$$

yielding an invariant non-degenerate symmetric bilinear form. We can identify $\hat{\mathbf{g}}''$ with the subalgebra $\mathbf{g}' \oplus \mathbb{C}h_{-1}$ of $\mathbf{g}$ by

$$\begin{gathered} x_{\alpha_i}(0) = e_i, \quad x_{-\alpha_i}(0) = -f_i, \quad h_i(0) = h_i, \qquad 1 \leqslant i \leqslant l, \\ x_{-\alpha^+}(1) = e_0, \qquad x_{\alpha^+}(-1) = -f_0, \qquad c - \alpha^+ = h_0, \\ -c - d = h_{-1}. \end{gathered} \tag{3.17}$$

We take

$$\mathbf{H} = \mathbf{h} \oplus \mathbb{C}c \oplus \mathbb{C}d \tag{3.18}$$

as a Cartan subalgebra of $\hat{\mathbf{g}}''$, and use the form $\langle\ ,\ \rangle$ to identify $\mathbf{H}$ with its dual space $\mathbf{H}^*$. Then the root system of $\hat{\mathbf{g}}''$ is

$$\hat{\Delta} = \{\alpha + nc, mc \mid \alpha \in \Delta, n \in \mathbb{Z}, 0 \neq m \in \mathbb{Z}\} \tag{3.19}$$

and we take

$$\alpha_0 = c - \alpha^+, \quad \alpha_1, ..., \alpha_l \tag{3.20}$$

as simple roots. The fundamental weights of $\hat{\mathbf{g}}''$, $\hat{\omega}_0, \hat{\omega}_1, ..., \hat{\omega}_l$, are determined up to multiples of c by the conditions

$$\langle \hat{\omega}_i, \alpha_j \rangle = \delta_{ij}, \qquad 0 \leqslant i, j \leqslant l. \tag{3.21}$$

This gives

$$\hat{\omega}_0 = d, \quad \hat{\omega}_i = n_i d + \omega_i, \qquad 1 \leqslant i \leqslant l, \tag{3.22}$$

where $n_i = \langle \omega_i, \alpha^+ \rangle$ so $\alpha^+ = \sum_{1 \leqslant i \leqslant l} n_i \alpha_i$. If we take the simple roots of $\mathbf{g}$ to be $\alpha_{-1} = -c - d, \alpha_0, ..., \alpha_l$, then the fundamental weights of $\mathbf{g}$, $\Omega_{-1}, \Omega_0, ..., \Omega_l$, determined uniquely by the conditions

$$\langle \Omega_i, \alpha_j \rangle = \delta_{ij}, \qquad -1 \leqslant i, j \leqslant l, \tag{3.23}$$

are

$$\Omega_{-1} = -c, \quad \Omega_0 = d - c, \quad \Omega_i = n_i(d - c) + \omega_i, \qquad 1 \leqslant i \leqslant l. \tag{3.24}$$

From (3.1) it follows that

$$\mathbf{g} = \mathbf{g}^- \oplus \mathbf{H} \oplus \mathbf{g}^+, \tag{3.25}$$

where

$$\mathbf{H} = \sum_{-1 \leqslant i \leqslant l} \mathbb{C} h_i \tag{3.26}$$

is a Cartan subalgebra for $\mathbf{g}$, and $\mathbf{g}^+$ is spanned by vectors of the form

$$[e_{i_1}, ..., e_{i_n}], \qquad n \geqslant 1, \tag{3.27}$$

and $\mathbf{g}^-$ is spanned by vectors of the form

$$[f_{i_1}, ..., f_{i_n}], \qquad n \geqslant 1. \tag{3.28}$$

If we decompose $\mathbf{g}$ into $\hat{\mathbf{g}}''$-modules, we get the $\mathbb{Z}$-grading

$$\mathbf{g} = \sum_{m \in \mathbb{Z}} \mathbf{g}_m, \tag{3.29}$$

where $\mathbf{g}_0 = \hat{\mathbf{g}}''$ and $\mathbf{g}_m$ is the span of the vectors (3.27) if $m > 0$, (3.28) if $m < 0$, having $|m|$ occurrences of the subscript -1. Since $[c, \hat{\mathbf{g}}''] = 0$, $[c, e_{-1}] = -e_{-1}$ and $[c, f_{-1}] = f_{-1}$, it follows that ad c acts as $-m$ on $\mathbf{g}_m$. The scalar by which c acts on an irreducible $\hat{\mathbf{g}}''$-module is called the level of the module.

The standard $\hat{\mathbf{g}}''$-modules are those irreducible $\hat{\mathbf{g}}''$-modules with dominant integral highest weight. Recall that $\Lambda = zc + \sum_{0 \leqslant i \leqslant l} m_i \hat{\omega}_i$ is called dominant integral when $0 \leqslant m_i \in \mathbb{Z}$ for $0 \leqslant i \leqslant l$ and $z \in \mathbb{C}$. We denote the standard $\hat{\mathbf{g}}''$-module with highest weight Λ by $V(\Lambda)$. $V(\hat{\omega}_i + zc)$ is called an ith fundamental module, and $V(\hat{\omega}_0 + zc)$ is called a basic module. The level of $V(\Lambda)$ is $\langle \Lambda, c \rangle = m_0 + n_1 m_1 + ... + n_l m_l$. For each standard $\hat{\mathbf{g}}''$-module

$V(\Lambda)$ one has a dual (contragredient) $\hat{\mathbf{g}}''$-module, $V(\Lambda)^*$, which is the irreducible $\hat{\mathbf{g}}''$-module with lowest weight $-\Lambda$.

Let $V = V(\hat{\omega}_0 + c)$ and $V^* = V(\hat{\omega}_0 + c)^*$. We can identify $\mathbf{g}_{-1}$ with V and $\mathbf{g}_1$ with V^* so that f_{-1} is a highest weight vector in V, and e_{-1} is a lowest weight vector in V^* (see [FF]). There is a unique non-degenerate invariant symmetric bilinear extension of the form $\langle\ ,\ \rangle$ from $\mathbf{g}_0 = \hat{\mathbf{g}}''$ to

$$\mathbf{g}_{\mathrm{loc}} = V \oplus \hat{\mathbf{g}}'' \oplus V^*, \tag{3.30}$$

and it satisfies

$$\langle e_{-1}, f_{-1} \rangle = 1. \tag{3.31}$$

The action of d on V and V^* is determined by the brackets

$$[h_{-1}, f_{-1}] = -2f_{-1}, \qquad [h_{-1}, e_{-1}] = 2e_{-1}, \tag{3.32}$$

which imply

$$[d, f_{-1}] = f_{-1}, \qquad [d, e_{-1}] = -e_{-1}. \tag{3.33}$$

So f_{-1} has weight $\hat{\omega}_0 + c = -\alpha_{-1}$ and e_{-1} has weight $-\hat{\omega}_0 - c = \alpha_{-1}$.

THEOREM 3.1. *Let $\bar{G}$ and J be constructed as in Section 2 from (3.30). Then $\bar{G}/J$ is isomorphic to the simple Lorentzian Kac–Moody algebra $\mathbf{g}$, $J^- = [\bar{G}, J_{-2}]$ is generated by $[f_{-1}, [f_{-1}, f_0]]$ and J_{-2} is an irreducible standard $\hat{\mathbf{g}}''$-module with highest weight $c + 2d + \alpha^+$. Furthermore, $J^+ = [\bar{G}, J_2]$ is generated by $[e_{-1}, [e_{-1}, e_0]]$ and J_2 is an irreducible lowest weight $\hat{\mathbf{g}}''$-module with lowest weight $-c - 2d - \alpha^+$.*

Proof. It is clear that all the defining relations (3.1) are valid in $\bar{G}$ except

$$[f_{-1}, [f_{-1}, f_0]] = 0, \qquad [e_{-1}, [e_{-1}, e_0]] = 0. \tag{3.34}$$

From $[e_i, f_j] = \delta_{ij} h_j$, $[h_i, e_j] = A_{ij} e_j$ and $[h_i, f_j] = -A_{ij} f_j$ we get

$$[e_i, [f_{-1}, [f_{-1}, f_0]]] = 0 \qquad \text{for} \quad -1 \leqslant i \leqslant l \tag{3.35}$$

and

$$[f_i, [e_{-1}, [e_{-1}, e_0]]] = 0 \qquad \text{for} \quad -1 \leqslant i \leqslant l, \tag{3.36}$$

which imply that

$$[f_{-1}, [f_{-1}, f_0]] \in J_{-2} \tag{3.37}$$

and

$$[e_{-1}, [e_{-1}, e_0]] \in J_2. \tag{3.38}$$

Therefore, all of the relations (3.1) are valid in $\bar{G}/J$, which must be isomorphic to the simple Lorentzian Kac–Moody algebra $\mathbf{g}$ by the Gabber–Kac theorem [GK] (see also [KMW, Proposition 3.1]). This means that $J^- = [\bar{G}, J_{-2}]$ is generated by (3.37) and J_{-2} is an irreducible standard $\hat{\mathbf{g}}''$-module with highest weight $c + 2d + \alpha^+$. Similarly, we get that $J^+ = [\bar{G}, J_2]$ is generated by (3.38) and J_2 is an irreducible lowest weight $\hat{\mathbf{g}}''$-module with lowest weight $-c - 2d - \alpha^+$. ∎

Let us now turn to the construction of modules for this Lorentzian Kac–Moody algebra $\mathbf{g}$. Let U be an irreducible standard highest weight $\hat{\mathbf{g}}''$-module with highest weight vector u_0 of weight $\Lambda = \sum_{-1 \leqslant i \leqslant l} m_i \Omega_i$, $0 \leqslant m_i \in \mathbb{Z}$, and let H be the $\bar{G}$-module as defined in Section 2 from V and U. Let $\mathscr{U}(\bar{G})$ denote the universal enveloping algebra of $\bar{G}$, and note that $H = \mathscr{U}(\bar{G}) \cdot u_0$ is a standard cyclic $\bar{G}$-module. Identifying J_{-2} as a subspace of $V \wedge V$, let $W = J_{-2}$ define the $\bar{G}$-submodule K as in Proposition 2.5. Then H/K is a $\mathbf{g}$-module, $\mathbb{Z}$-graded as in (2.40). Each graded piece of H/K is a quotient of a finite tensor product of highest weight $\hat{\mathbf{g}}''$-modules, so each f_i, $0 \leqslant i \leqslant l$, acts locally nilpotently on H/K. From (2.13) it is clear that f_{-1} does not act locally nilpotently. From the formula

$$e_{-1} f_{-1}^{k+1} \cdot u_0 = -(k+1)(k - m_{-1}) f_{-1}^{k} \cdot u_0, \tag{3.39}$$

which is valid in $\mathscr{U}(\bar{G})$ for $k \geqslant 0$, we see that

$$u_{\max} = f_{-1}^{m_{-1}+1} \cdot u_0 \tag{3.40}$$

is a maximal vector in H. Let

$$N = \mathscr{U}(\bar{G}) \cdot u_{\max}. \tag{3.41}$$

Then N is a proper non-trivial $\bar{G}$-submodule of H. Let $M = K + N$.

LEMMA 3.2. *On the* $\mathbf{g}$*-module* H/M *the action of* f_i *is locally nilpotent for* $-1 \leqslant i \leqslant l$.

Proof. Since M contains K, H/M is a $\mathbf{g}$-module. The only point to prove is that f_{-1} acts locally nilpotently on H/M. In $\mathscr{U}(\bar{G})$ we have the relations

$$f_{-1} f_i = f_i f_{-1} \qquad \text{for} \quad 1 \leqslant i \leqslant l \tag{3.42}$$

and

$$f_{-1}^2 f_0 = (f_0 f_{-1} + 2[f_{-1}, f_0]) f_{-1} + [f_{-1}, [f_{-1}, f_0]]. \qquad (3.43)$$

Any vector in H can be written as a linear combination of vectors of the form

$$h = f_{i_1} \cdot f_{i_2} \cdot \ldots \cdot f_{i_r} \cdot u_0 \qquad \text{for} \quad -1 \leqslant i_1, \ldots, i_r \leqslant l, \qquad (3.44)$$

so for p sufficiently large, (3.42) and (3.43) imply that $f_{-1}^p \cdot h \in M$. ∎

We may now apply a result of Kac [K2, Sect. 2, Theorem 2] to show that our construction has produced a highest weight irreducible $\mathbf{g}$-module, $\mathfrak{V}(\Lambda)$, for arbitrary dominant integral weight Λ.

THEOREM 3.3. *The* $\mathbf{g}$*-module* H/M *is an irreducible highest weight module with highest weight* Λ.

Proof. Since H is a standard cyclic $\bar{G}$-module, H/K is a standard cyclic $\mathbf{g}$-module. Therefore, H/K is a homomorphic image of a $\mathbf{g}$ Verma module $\mathscr{V}(\Lambda)$ with highest weight generating vector v_0. The maximal proper submodule $\mathscr{I}$ of $\mathscr{V}(\Lambda)$ is generated by the vectors

$$f_i^{m_i+1} \cdot v_0 \qquad \text{for} \quad -1 \leqslant i \leqslant l. \qquad (3.45)$$

The images of these vectors in H/K,

$$f_i^{m_i+1} \cdot u_0 + K, \qquad (3.46)$$

are zero for $0 \leqslant i \leqslant l$ since U is an irreducible highest weight $\hat{\mathbf{g}}''$-module with highest weight Λ. Composing with the projection onto $H/M \cong (H/K)/(M/K)$, they are all zero, so the maximal proper ideal equals the kernel of the map onto H/M, giving the isomorphism $H/M \cong \mathscr{V}(\Lambda)/\mathscr{I} \cong \mathfrak{V}(\Lambda)$. ∎

The $\mathbb{Z}$-grading of H in (2.12) induced the grading of K and H/K in Proposition 2.5, and induces such a grading on N, M, M/K and H/M. For example,

$$N_{-n} = 0 \qquad \text{for} \quad 0 \leqslant n \leqslant m_{-1} \qquad (3.47)$$

and

$$N_{-(m_{-1}+1)} = \mathscr{U}(\hat{\mathbf{g}}'') \cdot u_{\max} \cong V(\Lambda - (m_{-1}+1)\,\alpha_{-1}). \qquad (3.48)$$

We finish by giving precise descriptions of the first few graded pieces of the irreducible $\mathbf{g}$-module H/M constructed above.

COROLLARY 3.4. *The irreducible* $\mathbf{g}$*-module* H/M *with highest weight* $\Lambda = \sum_{-1 \leqslant i \leqslant l} m_i \Omega_i$ *has the* $\mathbb{Z}$*-grading*

$$H/M = (H/M)_0 \oplus (H/M)_{-1} \oplus (H/M)_{-2} \oplus \dots$$

and we have $(H/M)_0 = U = V(\Lambda)$.

(a) *If* $m_{-1} = 0$ *then we have*

$$(H/M)_{-1} = \frac{V \otimes U}{N_{-1}} = \frac{V(\hat{\omega}_0 + c) \otimes V(\Lambda)}{V(\hat{\omega}_0 + c + \Lambda)}.$$

(b) *If* $m_{-1} = 1$ *then we have*

$$(H/M)_{-1} = V \otimes U = V(\hat{\omega}_0 + c) \otimes V(\Lambda)$$

and

$$(H/M)_{-2} = \frac{V \otimes V \otimes U}{N_{-2} \oplus K_{-2}} = \frac{V(\hat{\omega}_0 + c) \otimes V(\hat{\omega}_0 + c) \otimes V(\Lambda)}{V(2\hat{\omega}_0 + 2c) \otimes V(\Lambda) \oplus V(2\hat{\omega}_0 + 2c - \alpha_0) \otimes V(\Lambda)}.$$

(c) *If* $m_{-1} \geqslant 2$ *then we have*

$$(H/M)_{-1} = V \otimes U = V(\hat{\omega}_0 + c) \otimes V(\Lambda)$$

and

$$(H/M)_{-2} = \frac{V \otimes V \otimes U}{K_{-2}} = \frac{V(\hat{\omega}_0 + c) \otimes V(\hat{\omega}_0 + c) \otimes V(\Lambda)}{V(2\hat{\omega}_0 + 2c - \alpha_0) \otimes V(\Lambda)}.$$

COROLLARY 3.5. *If* $\Lambda = m_{-1} \Omega_{-1}$ *then* $\dim(U) = 1$ *and in each part of Corollary* 3.4 *we may replace* $V(\mu) \otimes V(\Lambda)$ *by* $V(\mu + \Lambda)$ *for any* μ.

We now apply the above results to the fundamental representations.

COROLLARY 3.6. *Let* $\mathbf{g}_0$ *be the affine subalgebra of the Lorentzian Kac–Moody algebra* $\mathbf{g}$ *as in Theorem* 3.1. *Then* $H/M \cong \mathfrak{V}(\Omega_i)$ *for* $-1 \leqslant i \leqslant l$ *is a fundamental* $\mathbf{g}$*-module.*

(a) *If* $\Lambda = \Omega_{-1} = -c$ *then*

$$(H/M)_0 = U = V(-c)$$

is a trivial 1*-dimensional* $\mathbf{g}_0$*-module,*

$$(H/M)_{-1} = V(\hat{\omega}_0)$$

is a basic $\mathfrak{g}_0$*-module,*

$$(H/M)_{-2} = \frac{V(\hat{\omega}_0 + c) \otimes V(\hat{\omega}_0)}{V(2\hat{\omega}_0 + c) \oplus V(2\hat{\omega}_0 + c - \alpha_0)}$$

is essentially the tensor square of a basic $\mathfrak{g}_0$*-module modulo the sum of the top symmetric and top antisymmetric irreducible components.*

(b) *If* $\Lambda = \Omega_0 = \hat{\omega}_0 - c$ *then*

$$(H/M)_0 = U = V(\hat{\omega}_0 - c)$$

is a basic $\mathfrak{g}_0$*-module,*

$$(H/M)_{-1} = \frac{V(\hat{\omega}_0 + c) \otimes V(\hat{\omega}_0 - c)}{V(2\hat{\omega}_0)}$$

is essentially the tensor square of a basic $\mathfrak{g}_0$*-module modulo the top symmetric irreducible component.*

(c) *If* $\Lambda = \Omega_i = \hat{\omega}_i - n_i c$, $1 \leqslant i \leqslant l$, *then*

$$(H/M)_0 = U = V(\hat{\omega}_i - n_i c)$$

is a fundamental $\mathfrak{g}_0$*-module,*

$$(H/M)_{-1} = \frac{V(\hat{\omega}_0 + c) \otimes V(\hat{\omega}_i - n_i c)}{V(\hat{\omega}_0 + \hat{\omega}_i - (n_i - 1)c)}$$

is essentially the tensor product of a basic and a fundamental $\mathfrak{g}_0$*-module modulo the top irreducible component.*

A generating function formula for the level 2 root multiplicities for the hyperbolic algebra of type $\hat{A}_1^{(1)}$ was given in [FF]. The same information for the hyperbolic algebra of type $E_{10} = \hat{E}_8^{(1)}$ was given in [KMW]. Using the same methods, we can now give generating function formulas for the level 2 weight multiplicities of the three fundamental modules for $\hat{A}_1^{(1)}$. Such formulas for some of the fundamental E_{10}-modules would follow from character formulas for certain $E_8^{(1)}$-modules.

Let $\phi(q) = \prod_{n \geqslant 1} (1 - q^n)$ and define the generating functions

$$\sum_{k \geqslant 0} M_{-1}(k)\, q^k = \frac{\phi(q^2)\, q^{-3}}{\phi(q)\, \phi(q^4)} \left[\prod_{n \geqslant 1} (1 + q^{2n-1}) - 1 - q \right], \tag{3.49}$$

$$\sum_{k \geqslant 0} M_0(k)\, q^k = \frac{\phi(q^2)\, q^{-1}}{\phi(q)\, \phi(q^4)} \left[\prod_{n \geqslant 1} (1 + q^{2n-1}) - 1 \right], \tag{3.50}$$

$$\sum_{k \geqslant 0} M_1(k)\, q^k = \frac{\phi(q^2)\, q^{-1}}{\phi(q)^2} \left[\prod_{n \geqslant 1} (1 + q^n) - 1 \right]. \tag{3.51}$$

Let $\mathbf{g}$ be of type $\hat{A}_1^{(1)}$, so $l=1$ and $n_1=1$ in (3.24). As was shown in [FF], a weight $\Lambda = m_{-1}\Omega_{-1} + m_0\Omega_0 + m_1\Omega_1$ can be associated with a 2×2 symmetric matrix

$$L = \begin{bmatrix} m_{-1}+m_0+m_1 & \frac{1}{2}m_1 \\ \frac{1}{2}m_1 & m_0+m_1 \end{bmatrix} \tag{3.52}$$

so that $\langle \Lambda, \Lambda \rangle = -2\det(L)$ and w in the Weyl group $W \cong PGL_2(\mathbb{Z})$ acts on L by $w \cdot L = wLw^t$. It follows that

$$\mu = \begin{bmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & 2 \end{bmatrix} \tag{3.53}$$

is a level 2 weight of $\mathfrak{V}(\Omega_i)$, $i=-1, 0, 1$, when $\det(\mu) \geqslant 1$, $a, b \in \mathbb{Z}$, and b is odd if $i=1$ but even otherwise.

COROLLARY 3.7. *With notation as above, the multiplicity of a level 2 weight μ in a fundamental $\hat{A}_1^{(1)}$-module $\mathfrak{V}(\Omega_i)$, is $M_i(\det(\mu)-1)$ if $i=-1$, 0, but it is $M_1(\frac{1}{2}\det(\mu)-\frac{7}{8})$ if $i=1$.*

REFERENCES

[B1] R. E. BORCHERDS, Generalized Kac–Moody algebras, *J. Algebra* **115** (1988), 501–512.

[B2] R. E. BORCHERDS, Monstrous moonshine and monstrous Lie superalgebras, *Invent. Math.* **109** (1992), 405–444.

[FF] A. J. FEINGOLD AND I. B. FRENKEL, A hyperbolic Kac–Moody algebra and the theory of Siegel modular forms of genus 2, *Math. Ann.* **263** (1983), 87–144.

[Fr1] I. FRENKEL, Representations of Kac–Moody algebras and dual resonance models, *in* "Applications of Group Theory in Physics and Mathematical Physics," pp. 325–353, Lectures in Applied Mathematics, Vol. 21, Amer. Math. Soc., Providence, RI, 1985.

[Fr2] I. FRENKEL, Beyond affine Lie algebras, *in* "Proceedings of the International Congress of Mathematicians, Berkeley, California, 1986," pp. 821–839, Amer. Math. Soc., Providence, RI, 1987.

[FK] I. FRENKEL AND V. G. KAC, Basic representations of affine Lie algebras and dual resonance models, *Invent. Math.* **62** (1980), 23–66.

[GB] O. GABBER AND V. G. KAC, On defining relations of certain infinite dimensional Lie algebras, *Bull. Amer. Math. Soc.* **5** (1981), 185–189.

[G] K. GAWEDZKI, Conformal field theory, Séminaire Bourbaki, Exposé 704, *Astérisque*, **177–178** (1989), 95–126.

[H] HARISH-CHANDRA, Representations of semisimple Lie groups, IV, *Amer. J. Math.* **77** (1955), 743–777.

[K1] V. G. KAC, Simple irreducible graded Lie algebras of finite growth, *Izv. Akad. Nauk. SSSR* **32** (1968), 1323–1367 [in Russian]; English transl., *Math. USSR-Izv.* **2** (1968), 1271–1311.

[K2] V. G. KAC, Infinite-dimensional algebras, Dedekind's η-function, classical Mobius function and the very strange formula, *Adv. Math.* **30** (1978), 85–136.

[KMW] V. KAC, R. V. MOODY, AND M. WAKIMOTO, On E_{10}, *in* "Differential Geometrical Methods in Theoretical Physics" (K. Bleuler and M. Werner, Eds.), pp. 109–128, Kluwer Academic, 1988.

[Ka1] S.-J. KANG, Construction of Kac–Moody algebras by standard gradation, preprint, 1989.

[Ka2] S.-J. KANG, Kac–Moody Lie algebras, spectral sequences, and the Witt formula, Transactions of the A.M.S., to appear.

[Kan] I. L. KANTOR, Graded Lie algebras, *Trudy Sem. Vektor. Tenzor. Anal.* **15** (1970), 227–266. [in Russian]

[Mat] O. MATHIEU, Formules de caractères pour les algèbres de Kac–Moody générales, *Astérisque* **159–160** (1988), 1–267.

[Mo] R. V. MOODY, A new class of Lie algebras, *J. Algebra* **10** (1968), 210–230.