Contemporary Mathematics

Fusion Rules for Affine Kac-Moody Algebras

Alex J. Feingold

ABSTRACT. This is an expository introduction to fusion rules for affine Kac-Moody algebras, with major focus on the algorithmic aspects of their computation and the relationship with tensor product decompositions. Many explicit examples are included with figures illustrating the rank 2 cases. New results relating fusion coefficients to tensor product coefficients are proved, and a conjecture is given which shows that the Frenkel-Zhu affine fusion rule theorem can be seen as a beautiful generalization of the Parasarathy-Ranga Rao-Varadarajan tensor product theorem. Previous work of the author and collaborators on a different approach to fusion rules from elementary group theory is also explained.

1. Introduction

Fusion rules play a very important role in conformal field theory [**Fu**], in the representation theory of vertex operator algebras [**FLM, FHL, FZ**], and in quite a few other areas. This paper is not meant to be comprehensive, but should be a useful introduction to the subject, with major focus on the algorithmic aspects of computing fusion rules in the case of affine Kac-Moody algebras. I have included many explicit examples and figures illustrating the rank 2 cases which can be done graphically on a sheet of paper. The Kac-Walton algorithm [**Kac, Wal**] for fusion coefficients is closely related to the Racah-Speiser algorithm for tensor product decompositions, which was the subject of my thesis [**F1, F2**]. I have included here some discussion of this relationship and some implications of my thesis for the computation of fusion coefficients. In Theorems 6.1 and 6.2, for fixed dominant integral weights λ and μ, I determine the values of level k for which all tensor product multiplicities, $Mult^{\nu}_{\lambda,\mu}$, are equal to the corresponding level k fusion coefficients, $N^{(k)\ \nu}_{\lambda,\mu}$, for all dominant integral ν. I have recalled the results of Parasarathy, Ranga Rao and Varadarajan [**PRV**] on tensor product multiplicities in Theorem 5.2 , and the results of Frenkel and Zhu [**FZ**] on fusion coefficients in Theorem 6.3. I have included a conjecture on fusion coefficients which I believe is a restatement of the

2000 *Mathematics Subject Classification.* Primary 17B67, 17B65, 81T40; Secondary 81R10, 05E10.

Key words and phrases. Fusion Rules, Affine Kac-Moody Algebras.

I wish to express my thanks to Prof. N. Sthanumoorthy and the other organizers of the Symposium for inviting me, and for their generous hospitality.

Frenkel-Zhu theorem in a form which shows it to be a beautful generalization of the PRV theorem. In joint work [**AFW, FW**] we have tried to understand fusion rules from a combinatorial point of view which is quite different from the approaches of others [**BMW, BKMW, T**]. The idea for our new approach was inspired by our work on explicit spinor constructions [**FFR, FRW**]. In [**AFW**] we explained all of the (p, q)-minimal model fusion rules [**Wa**] from elementary 2-groups. The (p, q)-minimal models are a certain series of highest weight representations of the Virasoro algebra [**KR**] which also have the structure of a vertex operator algebra [**FLM**], and modules for it [**FZ**]. In [**FW**] we explained the fusion rules for all positive integral levels for type A_n affine Kac-Moody algebras if $n = 1$ or $n = 2$. That work is explained in this paper.

This paper is an expanded version of two lectures I presented at the Ramanujan International Symposium on Kac-Moody Lie Algebras and Applications, Jan. 28 - 31, 2002, Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai, India. It was a great honor to be invited to this symposium, and I was pleased to be able to include a connection of my work with some work of Ramanujan, whose genius continues to inspire great mathematics all around the world.

2. Definition of Fusion Algebra

Let us begin with the definition of fusion algebra given by J. Fuchs [**Fu**].

A fusion algebra F is a finite dimensional commutative associative algebra over $\mathbf{Q}$ with some basis

$$B = \{x_a \mid a \in A\}$$

so that the structure constants $N_{a,b}^c$ defined by

$$x_a \cdot x_b = \sum_{c \in A} N_{a,b}^c x_c$$

are non-negative integers. There must be a distinguished index $\Omega \in A$ with the following properties. Define a matrix

$$C = [C_{a,b}] = [N_{a,b}^{\Omega}]$$

and define an associated "conjugation" map $\mathcal{C} : F \to F$ by

$$\mathcal{C}(x_a) = \sum_{b \in A} C_{a,b} x_b.$$

It is required that $\mathcal{C}$ be an involutive automorphism of F, so $\mathcal{C}^2 = I_F$ and $C^2 = I$. Because $0 \leq N_{a,b}^c \in \mathbf{Z}$, either $C = I$ or C must be an order 2 permutation matrix, that is, there is a permutation $\sigma : A \to A$ with $\sigma^2 = 1$ and

$$C_{a,b} = \delta_{a,\sigma(b)}.$$

Since $\mathcal{C}$ is an automorphism, we must also have

$$\mathcal{C}(x_a) \cdot \mathcal{C}(x_b) = \mathcal{C}(x_a \cdot x_b),$$

that is,

$$x_{\sigma(a)} \cdot x_{\sigma(b)} = \sum_{c \in A} N_{a,b}^c x_{\sigma(c)}$$

which means that

$$N_{\sigma(a),\sigma(b)}^{\sigma(c)} = N_{a,b}^c.$$

Write $\sigma(a) = a^+$ and call x_{a^+} the conjugate of x_a. Use it to define the non-negative integers

$$N_{a,b,c} = N_{a,b}^{c^+}$$

which, by commutativity and associativity of the algebra product, are completely symmetric in a, b and c. To see this, note that commutativity means $N_{a,b}^c = N_{b,a}^c$ for all $a, b, c \in A$, so $N_{a,b,c} = N_{b,a,c}$. Associativity means

$$\sum_{d \in A} N_{a,b}^d \, N_{d,c}^e = \sum_{d \in A} N_{b,c}^d \, N_{a,d}^e$$

for all $a, b, c, e \in A$. Taking $e = \Omega$ and using $N_{a,b}^{\Omega} = \delta_{a,b^+} = \delta_{a^+,b}$, this gives $N_{a,b}^{c^+} = N_{b,c}^{a^+}$, so $N_{a,b,c} = N_{b,c,a}$. This order 3 cyclic permutation and the transposition switching a and b generate all permutations of a, b and c. Using this we also find

$$\begin{aligned} N_{\Omega,b}^c &= N_{\Omega,b,c^+} = N_{b,c^+,\Omega} = N_{b,c^+}^{\Omega^+} \\ &= N_{b^+,c}^{\Omega} = C_{b^+,c} = \delta_{b,c} \end{aligned}$$

which means x_Ω is a multiplicative identity element in F, so we write $x_\Omega = 1$. It also follows that $\Omega^+ = \Omega$.

Here are some examples of fusion algebras, which we will later see come from representations of affine Kac-Moody algebras of some "level". The algebras are presented by giving a table of products of the basis elements. These fusion rule tables were produced by the computer program of Bert Schellekens, called "Kac", available from his webpage: http://norma.nikhef.nl/~t58/

3. Examples of Fusion Algebras

Table 1: Fusion Table for A_1 of level $k = 2$

[i]·[j]	[0]	[1]	[2]
[0]	[0]	[1]	[2]
[1]		[0]	[2]
[2]			[0]+[1]

In this example, we have

$$A = \{\Omega = 0, 1, 2\}, \qquad B = \{x_0 = [0], x_1 = [1], x_2 = [2]\}.$$

From Table 1 we see that, for example,

$$[2] \cdot [2] = 1[0] + 1[1] + 0[2]$$

and we can read off particular structure constants, for example,

$$N_{2,2}^0 = 1 \quad N_{2,2}^1 = 1, \quad N_{2,2}^2 = 0.$$

It is also easy to see that

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We would get the same fusion table for B_2 of level $k = 1$.

Table 2: Fusion Table for A_1 of level $k = 3$

[i]·[j]	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]		[0]	[3]	[2]
[2]			[0]+[2]	[1]+[3]
[3]				[0]+[2]

In this example, we have

$$A = \{\Omega = 0, 1, 2, 3\}, \qquad B = \{[0], [1], [2], [3]\},$$
$$[2] \cdot [3] = 0[0] + 1[1] + 0[2] + 1[3],$$
$$N^0_{i,j} = \delta_{i,j} \text{ so } C = I_4.$$

Table 3: Fusion Table for A_2 of level $k = 2$

[i]·[j]	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]		[2]	[0]	[4]	[5]	[3]
[2]			[1]	[5]	[3]	[4]
[3]				[0]+[3]	[1]+[4]	[2]+[5]
[4]					[2]+[5]	[0]+[3]
[5]						[1]+[4]

In this example, we have

$$A = \{\Omega = 0, 1, 2, 3, 4, 5\}, \qquad B = \{[0], [1], [2], [3], [4], [5]\},$$
$$[1] \cdot [2] = [0] = [0] \cdot [0], \qquad [3] \cdot [3] = [0] + [3] = [4] \cdot [5],$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Table 4: Partial fusion Table for A_2 of level $k = 3$

[i]·[j]	[0]	[1]	[2]	[9]
[0]	[0]	[1]	[2]	[9]
[1]		[2]	[0]	[9]
[2]			[1]	[9]
[9]				[0]+[1]+[2]+2[9]

In this example, we have

$$A = \{\Omega = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, \qquad B = \{[0], [1], [2], \cdots, [9]\},$$
$$[9] \cdot [9] = [0] + [1] + [2] + 2[9].$$

Note that this is the first example where a coefficient exceeds 1: $N^9_{9,9} = 2$.

Here is the fusion table for the affine algebra of type B_2 of level 2.

Table 5: Fusion Table for B_2 of level $k = 2$

[i]·[j]	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]		[0]	[3]	[2]	[4]	[5]
[2]			[0]+[4]+[5]	[1]+[4]+[5]	[2]+[3]	[2]+[3]
[3]				[0]+[4]+[5]	[2]+[3]	[2]+[3]
[4]					[0]+[1]+[5]	[4]+[5]
[5]						[0]+[1]+[4]

4. Notations

Now we will introduce notations and discuss how fusion algebras are associated with representations of untwisted affine Kac-Moody algebras of fixed level. Let $\mathbf{g}$ be a finite dimensional simple Lie algebra of rank $N-1$ with Cartan matrix $A = [a_{ij}]$, and let

$$\hat{\mathbf{g}} = \mathbf{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}c \oplus \mathbf{C}d$$

be the corresponding affine algebra with derivation $d = -t\frac{d}{dt}$ adjoined as usual. Let H be the Cartan subalgebra of $\mathbf{g}$ and let

$$\mathcal{H} = H \oplus \mathbf{C}c \oplus \mathbf{C}d$$

be the Cartan subalgebra of $\hat{\mathbf{g}}$. The simple roots and the fundamental weights of $\mathbf{g}$ are linear functionals

$$\alpha_1, \cdots, \alpha_{N-1} \quad \text{and} \quad \lambda_1, \cdots, \lambda_{N-1},$$

respectively, in the dual space H^*. Let the integral weight lattice P be the $\mathbf{Z}$-span of the fundamental weights, and let

$$P^+ = \{n_1\lambda_1 + \cdots + n_{N-1}\lambda_{N-1} \mid 0 \le n_1, \cdots, n_{N-1} \in \mathbf{Z}\}$$

be the set of dominant integral weights of $\mathbf{g}$, and let

$$\theta = \sum_{i=1}^{N-1} \ell_i \alpha_i$$

be the highest root of $\mathbf{g}$. The symmetric bilinear form $(\cdot, \cdot)$ on H^* is determined by

$$a_{ij} = \langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \quad 1 \le i, j \le N-1$$

and the normalization $(\theta, \theta) = 2$. The fundamental weights are determined by the conditions $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for $1 \le i, j \le N-1$, and the special "Weyl vector"

$$\rho = \sum_{i=1}^{N-1} \lambda_i$$

will play an important role in several formulas. It is useful to define

$$\check{\lambda} = \frac{2\lambda}{(\lambda, \lambda)} \quad \text{for any } 0 \ne \lambda \in H^*,$$

so we can write $(\lambda_i, \check{\alpha}_j) = \delta_{ij}$ and $a_{ij} = (\alpha_i, \check{\alpha}_j)$. We may also express

$$\theta = \sum_{i=1}^{N-1} \check{\ell}_i \check{\alpha}_i \qquad \text{so} \qquad \check{\ell}_i = \frac{\ell_i(\alpha_i, \alpha_i)}{2}.$$

The dual Coxeter number of $\mathbf{g}$ is defined to be

$$\check{h} = 1 + \sum_{i=1}^{N-1} \check{\ell}_i = 1 + \langle \rho, \theta \rangle.$$

The Weyl group W of $\mathbf{g}$ is defined to be the group of endomorphisms of H^* generated by the simple reflections corresponding to the simple roots,

$$r_i(\lambda) = \lambda - (\lambda, \check{\alpha}_i), \qquad 1 \le i \le N-1.$$

This is a finite group of isometries which preserve P. There is a partial order defined on H^* defined by

$$\lambda \le \mu \quad \text{if and only if} \quad \mu - \lambda = \sum_{i=1}^{N-1} k_i \alpha_i \quad \text{for some } 0 \le k_i \in \mathbf{Z}.$$

For $\lambda \in P^+$ let V^λ denote the finite dimensional irreducible $\mathbf{g}$-module with highest weight λ. It has the weight space decomposition $V^\lambda = \bigoplus_{\beta \in H^*} V^\lambda_\beta$, where

$$V^\lambda_\beta = \{ v \in V^\lambda \mid h \cdot v = \beta(h) v, \forall h \in H \}$$

is the β weight space of V^λ. Of course, there are only finitely many $\beta \in H^*$ such that V^λ_β is nonzero, and we denote by Π^λ that finite set of such β. The dual space $(V^\lambda)^* = Hom(V^\lambda, \mathbf{C})$ is also an irreducible highest weight $\mathbf{g}$-module, called the contragredient module of V^λ. The action of $\mathbf{g}$ on $(V^\lambda)^*$ is given by

$$(x \cdot f)(v) = -f(x \cdot v) \quad \text{for} \quad x \in \mathbf{g},\ f \in (V^\lambda)^*,\ v \in V^\lambda.$$

The highest weight of $(V^\lambda)^*$ is denoted by $\lambda^+ = \lambda^*$, and equals the negative of the lowest weight of V^λ. For example, in the case when $\mathbf{g}$ is of type A_{N-1}, if $\lambda = \sum_{i=1}^{N-1} n_i \lambda_i$ then $\lambda^+ = \sum_{i=1}^{N-1} n_{N-i} \lambda_i$.

The simple roots and the fundamental weights of $\hat{\mathbf{g}}$ are linear functionals

$$\alpha_0, \alpha_1, \cdots, \alpha_{N-1} \quad \text{and} \quad \Lambda_0, \Lambda_1, \cdots, \Lambda_{N-1},$$

respectively, in the dual space $\mathcal{H}^*$. The simple roots of $\mathbf{g}$ form a basis of H^* (as do the fundamental weights), and we identify them with linear functionals in $\mathcal{H}^*$ having the same values on $H \subseteq \mathcal{H}$ and being zero on c and d. Let c^* and d^* in $\mathcal{H}^*$ be the functionals which are zero on H and which satisfy

$$c^*(c) = 1, \quad c^*(d) = 0, \quad d^*(c) = 0, \quad d^*(d) = 1.$$

Extend the bilinear form $(\cdot, \cdot)$ to $\mathcal{H}^*$ by letting

$$(c^*, H^*) = 0 = (d^*, H^*), \qquad (c^*, c^*) = 0 = (d^*, d^*), \text{ and } (c^*, d^*) = 1.$$

Then $\alpha_0 = d^* - \theta$ and

$$\Lambda_0 = c^*, \quad \Lambda_i = \ell_i \frac{(\alpha_i, \alpha_i)}{2}\, c^* + \lambda_i = \check{\ell}_i\, c^* + \lambda_i, \quad 1 \le i \le N-1,$$

are determined by the conditions $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$ for $0 \leq i, j \leq N-1$. Let the integral weight lattice $\hat{P}$ be the $\mathbf{Z}$-span of the fundamental weights, and let

$$\hat{P}^+ = \{\sum_{i=0}^{N-1} n_i \Lambda_i \mid 0 \leq n_i \in \mathbf{Z}\}$$

be the set of dominant integral weights of $\hat{\mathbf{g}}$. The affine Weyl group $\widehat{W}$ of $\hat{\mathbf{g}}$ is the group of endomorphisms of $\mathcal{H}^*$ generated by the simple reflections corresponding to the simple roots,

$$r_i(\Lambda) = \Lambda - (\Lambda, \check{\alpha}_i)\alpha_i, \qquad 0 \leq i \leq N-1.$$

This is an infinite group of isometries which preserve $\hat{P}$. The canonical central element, $c \in \hat{\mathbf{g}}$ acts on an irreducible $\hat{\mathbf{g}}$-module as a scalar k, called the level of the module. We will only discuss modules with highest weight $\Lambda \in \hat{P}^+$, which are the "nicest" in that they have affine Weyl group symmetry and satisfy the Weyl-Kac character formula. An irreducible highest weight $\hat{\mathbf{g}}$-module is uniquely determined by its highest weight

$$\Lambda = \sum_{i=0}^{N-1} n_i \Lambda_i \in \hat{P}^+$$

and, if we define $\ell_0 = 1 = \check{\ell}_0$, then

$$k = \Lambda(c) = \sum_{i=0}^{N-1} n_i \Lambda_i(c) = \sum_{i=0}^{N-1} n_i \ell_i \frac{(\alpha_i, \alpha_i)}{2} = \sum_{i=0}^{N-1} n_i \check{\ell}_i.$$

For fixed k there are only finitely many $\Lambda \in \hat{P}^+$ with $\Lambda(c) = k$, and we denote that finite set by $\hat{P}^+_k$. It is easy to see that $\widehat{W}$ preserves the level k weights $\{\Lambda \in \hat{P} \mid \Lambda(c) = k\}$. The affine hyperplane determined by the condition $\Lambda(c) = k$ can be projected onto H^* and the corresponding action of $\widehat{W}$ is such that the simple reflections r_i for $1 \leq i \leq N-1$ act as they were defined originally on H^*, as isometries generating the finite Weyl group W of $\mathbf{g}$. But the new affine reflection r_0 acts as $r_0(\lambda) = \lambda - (\lambda, \theta)\theta + k\theta = r_\theta(\lambda) + k\theta$, the composition of reflection r_θ and the translation by $k\theta$, which is not an isometry on H^*.

Irreducible $\hat{\mathbf{g}}$-modules $\hat{V}^\Lambda$ of level $k \geq 1$ are indexed by $\hat{P}^+_k$, but we can also index them by certain weights of $\mathbf{g}$ as follows. From the formulas above we can write

$$\Lambda = \sum_{i=0}^{N-1} n_i \Lambda_i = kc^* + \sum_{i=1}^{N-1} n_i \lambda_i.$$

So there is a bijection between $\hat{P}^+_k$ and the set of weights $\lambda = \sum_{i=1}^{N-1} n_i \lambda_i$ such that

$$k = n_0 + \sum_{i=1}^{N-1} n_i \ell_i \frac{(\alpha_i, \alpha_i)}{2} = n_0 + \sum_{i=1}^{N-1} n_i \check{\ell}_i = n_0 + \langle \lambda, \theta \rangle.$$

Since $n_0 \geq 0$, this is equivalent to the "level k condition"

$$\langle \lambda, \theta \rangle = \sum_{i=1}^{N-1} n_i \check{\ell}_i \leq k.$$

Define the set

$$P_k^+ = \{\lambda = \sum_{i=1}^{N-1} n_i\lambda_i \in P^+ \mid \langle \lambda, \theta \rangle \leq k\}$$

and let the index set A (as in the fusion algebra definition) be P_k^+. Then we see that irreducible modules on level k correspond to N-tuples of nonnegative integers

$$(n_0, n_1, \cdots, n_{N-1}) \text{ such that } k = \sum_{i=0}^{N-1} n_i \ell_i \frac{(\alpha_i, \alpha_i)}{2} = \sum_{i=0}^{N-1} n_i \check{\ell}_i.$$

Such an N-tuple corresponds to

$$\Lambda = kc^* + n_1\lambda_1 + \cdots + n_{N-1}\lambda_{N-1}.$$

Fix level $k \geq 1$ and write the fusion algebra product (which has not been defined yet!)

$$[\lambda] \cdot [\mu] = \sum_{\nu \in P_k^+} N_{\lambda,\mu}^{\nu} \; [\nu].$$

The distinguished identity element, $[0]$, corresponds to $\Lambda = kc^*$, and for each $[\lambda]$ there is a distinguished conjugate $[\lambda^+]$ such that $N_{\lambda,\mu}^0 = \delta_{\mu,\lambda^+}$. Knowing $N_{\lambda,\mu}^{\nu}$ is equivalent to knowing the completely symmetric coefficients

$$N_{\lambda,\mu,\nu} = N_{\lambda,\mu}^{\nu^+}.$$

Let $\mathcal{F}(\mathbf{g}, k)$ denote this fusion algebra.

In the case when $\mathbf{g} = sl_N$ is of type A_{N-1}, we have $\ell_i = 1$ and $(\alpha_i, \alpha_i) = 2$ for $0 \leq i \leq N-1$, so the set of all weights of level 1,

$$\hat{P}_1^+ = \{\Lambda_i \mid 0 \leq i \leq N-1\}$$

is precisely the set of the fundamental weights of $\hat{\mathbf{g}}$, and

$$P_1^+ = \{0, \lambda_i \mid 1 \leq i \leq N-1\}.$$

The level 1 fusion algebra $\mathcal{F}(sl_N, 1)$ has a basis $\{[0], [1], \cdots, [N-1]\}$ (in Schellekens notation, $[i]$ corresponds to λ_i for $1 \leq i \leq N-1$) and the fusion rules are given by the group $\mathbf{Z}_N$, the weight lattice modulo the root lattice of $\mathbf{g}$. This means

$$N_{\lambda,\mu}^{\nu} = \delta_{\lambda+\mu,\nu}$$

where the addition takes place in the quotient group of the weight lattice modulo the root lattice. So $\mathcal{F}(sl_N, 1)$ is the group algebra $\mathbf{Q}[\mathbf{Z}_N]$.

For $\mathbf{g} = so(2N)$, $N \geq 4$, of type D_N and rank N, we have $\ell_i = 1$ for $i = 0, 1, N-1, N$, $\ell_i = 2$ for $2 \leq i \leq N-2$, and $(\alpha_i, \alpha_i) = 2$ for $0 \leq i \leq N$, so the level 1 weights

$$\hat{P}_1^+ = \{\Lambda_i \mid i = 0, 1, N-1, N\}$$

are the fundamental weights corresponding to the four endpoints of the affine Dynkin diagram of $\hat{\mathbf{g}}$, and

$$P_1^+ = \{0, \lambda_i \mid i = 1, N-1, N\}$$

where λ_1 is the highest weight of the natural representation (of dimension $2N$), λ_{N-1} and λ_N are highest weights of the half-spinor representations (each of dimension 2^{N-1}. The group structure of the weight lattice modulo the root lattice is known to be the Klein 4-group $\mathbf{Z}_2 \times \mathbf{Z}_2$ if N is even, $\mathbf{Z}_4$ if N is odd, and P_1^+ is a set of coset representatives for that quotient group in either case. The fusion algebra for $\mathbf{g} = so(2N)$ on level 1 is then

$$\mathcal{F}(so(2N),1) = \begin{cases} \mathbf{Q}[\mathbf{Z}_2 \times \mathbf{Z}_2] & \text{if } N \text{ is even,} \\ \mathbf{Q}[\mathbf{Z}_4] & \text{if } N \text{ is odd.} \end{cases}$$

For $\mathbf{g} = so(2N+1)$, $N \geq 3$, of type B_N and rank N, we have $\ell_i = 1$ for $i = 0, 1$, $\ell_i = 2$ for $2 \leq i \leq N$, $(\alpha_i, \alpha_i) = 2$ for $0 \leq i \leq N-1$ and $(\alpha_N, \alpha_N) = 1$. In the special case of B_2, we have $(\ell_0, \ell_1, \ell_2) = (1,2,1)$, and $(\alpha_i, \alpha_i) = 2, 1, 2$ for $i = 0, 1, 2$, respectively. So for $N \geq 2$, the level 1 weights are

$$\hat{P}_1^+ = \{\Lambda_i \mid i = 0, 1, N\}.$$

The weight lattice modulo the root lattice is $\mathbf{Z}_2$.

For $\mathbf{g} = sp(2N)$, $N \geq 2$, of type C_N and rank N, we have $\ell_i = 1$ and $(\alpha_i, \alpha_i) = 2$ for $i = 0, N$, and $\ell_i = 2$ and $(\alpha_i, \alpha_i) = 1$ for $1 \leq i \leq N-1$. The weight lattice modulo the root lattice is $\mathbf{Z}_2$. The special case of $N = 2$ gives B_2.

We will not give further details about the exceptional algebras, but for $\mathbf{g}$ of type G_2 we should mention that $(\ell_0, \ell_1, \ell_2) = (1, 3, 2)$, where α_2 is the long root and $(\alpha_1, \alpha_1) = 2/3$.

5. Algorithms For Tensor Product Decompositions

There is a close relationship between the product in fusion algebras associated with an affine Kac-Moody algebra $\hat{\mathbf{g}}$ and tensor product decompositions of irreducible $\mathbf{g}$-modules. Let V^λ be the irreducible finite dimensional $\mathbf{g}$-submodule of $\hat{V}^\Lambda$ generated by a highest weight vector. In the special case when $\Lambda = k\Lambda_0 = kc$, that finite dimensional $\mathbf{g}$-module is V^0, the one dimensional trivial $\mathbf{g}$-module. Since $\mathbf{g}$ is semisimple, any finite dimensional $\mathbf{g}$-module is completely reducible. Therefore, we can write the tensor product of irreducible $\mathbf{g}$-modules

$$V^\lambda \otimes V^\mu = \sum_{\nu \in P^+} Mult_{\lambda,\mu}^\nu V^\nu$$

as the direct sum of irreducible $\mathbf{g}$-modules, including multiplicities. This decomposition is independent of the level k and is part of the basic representation theory of $\mathbf{g}$. The fusion products $[\lambda] \cdot [\mu]$ are obtained by a subtle truncation of the above summation.

The Racah-Speiser algorithm gives the formula

$$Mult_{\lambda,\mu}^\nu = \sum_{w \in W} \epsilon(w) Mult_\lambda(w(\nu + \rho) - \mu - \rho)$$

where W is the Weyl group of $\mathbf{g}$, $\epsilon(w) = (-1)^{length(w)}$ is the sign of w, the Weyl vector $\rho = \sum \lambda_i$ is the sum of the fundamental weights of $\mathbf{g}$, and $Mult_\lambda(\beta) = dim(V_\beta^\lambda)$ is the inner multiplicity of the weight β in V^λ. Recall that $\Pi^\lambda = \{\beta \in H^* \mid dim(V_\beta^\lambda) > 0\}$ denotes the set of all weights of V^λ.

In fact, the only weights ν for which $Mult_{\lambda,\mu}^\nu$ may be nonzero are those of the form $\nu = \beta + \mu$ where $\beta \in \Pi^\lambda$. This means the formula is a geometrical algorithm:

(1) Shift the weight diagram of V^λ by adding $\mu + \rho$.

(2) Use the Weyl group to move all shifted weights into the dominant chamber, where they accumulate as an alternating sum of inner multiplicities of V^λ, adding if the required w is even, subtracting if it is odd.

(3) The resulting pattern of numbers will be non-negative integers, zero if the shifted weight is on a chamber wall, and after shifting the pattern back by subtracting ρ, you will have the "outer" tensor product multiplicities.

This algorithm assumes that you can already produce the weight diagram of any irreducible module, V^λ, so we should have discussed that first, but in fact the special case of the Racah-Speiser algorithm when $\mu = 0$ gives a recursion for the inner multiplicities of V^λ. Since V^0 is the trivial one-dimensional module, $V^\lambda \otimes V^0 = V^\lambda$, so $Mult^\nu_{\lambda,0} = \delta_{\lambda,\nu}$ and therefore

$$0 = \sum_{w \in W} \epsilon(w) Mult_\lambda(w(\nu + \rho) - \rho)$$

for $\nu \neq \lambda$. One knows that $Mult_\lambda(w\lambda) = 1$ and $Mult_\lambda(w\nu) = Mult_\lambda(\nu)$ for all $w \in W$, so the above formula implies that

$$Mult_\lambda(\nu) = - \sum_{1 \neq w \in W} \epsilon(w) Mult_\lambda(\nu + \rho - w\rho)$$

for $\nu \neq \lambda$. Since $\rho > w\rho$ in the partial ordering on weights, this gives an effective recursion for $Mult_\lambda(\nu)$. It is instructive to carry out these recursions by hand in the rank 2 cases, where the geometry is simple to see on a sheet of paper. I have included in the appendices pages of type A_2, B_2 and G_2 weight lattices, including the reflecting axes, and pages with just the reflecting axes. If you make a copy of the former, you can put on it the weight diagram of a single irreducible module, V^λ, by starting with one dominant weight (make a heavy dot) at position λ. Then find all the dominant weights less than λ in the partial ordering. Apply the Weyl group to that set of weights to get all the weights of the module. An example of this for type A_2 with $\lambda = 3\lambda_1 + 2\lambda_2$ is given in the Appendix, Figure 8. To use the Racah recursion formula, make a copy of the reflection axes only on a transparency. (Choose the appropriate axis for the type of algebra from Figures 4 - 6.) Then place the weight diagram you made under the transparency, shifted by ρ. Using the shifted reflecting lines you can see the points which will be involved in the alternating sum for a given dominant ν in the diagram, and find the multiplicity of the ν weight space. Mark those multiplicities next to each weight, using the Weyl group action on the unshifted weight diagram to mark nondominant weights. Now you can use that marked weight diagram to compute the tensor product of that module with any other by the Racah-Speiser algorithm. You only need to put the diagram under the transparency of reflecting axes shifted by $\mu + \rho$ and follow steps (2) and (3) above. For example, using the weight diagram in Figure 8 to compute the tensor product decomposition of $V^\lambda \otimes V^\mu$ for $\lambda = 3\lambda_1 + 2\lambda_2$ and $\mu = \lambda_1$, one would see the shifted weight diagram shown in Figure 9, and find that the Racah-Speiser algorithm gives the answer

$$V^{4\lambda_1 + 2\lambda_2} \oplus V^{3\lambda_1 + \lambda_2} \oplus V^{2\lambda_1 + 3\lambda_2}.$$

But in that case, it would have been wiser to shift the weight diagram of V^{λ_1} by $\mu = 3\lambda_1 + 2\lambda_2$ plus ρ as shown in Figure 16. (Ignore for now the affine reflection line shown there.) The three weights of that fundamental module each have multiplicity 1, and after shifting by $4\lambda_1 + 3\lambda_2$, all of them are strictly inside the dominant chamber, so there are no cancellations and each of them gives a highest weight module in the tensor product decomposition as shown above.

Another method of recursively computing the weight multiplicities $Mult_\lambda(\nu)$ is as follows. Place a clear transparency over the weight lattice, locate the weights $w\rho = w(\lambda_1 + \lambda_2)$ for each $w \in W$, and make an open circle around each such point, large enough to see the underlying weight in the diagram. Since the differences between those points and the fixed point ρ is $\rho - w\rho$, if you rotate the transparency 180 degrees and place the point ρ over any weight ν of a weight diagram for V^λ, the other open circles of the transparency will lie over the points $\nu + \rho - w\rho$, which will be strictly above ν in the partial ordering of weights. The Racah recursion formula can then be implemented by taking the alternating sum of the multiplicities of those circled weights, assumed to have been already found by the initial data $Mult_\lambda(w\lambda) = 1$, or by the application of Weyl group symmetry $Mult_\lambda(w\beta) = Mult_\lambda(\beta)$ to multiplicities already found recursively. I have combined in Figure 7 the diagrams of the Weyl conjugates of ρ for each type. If you copy this page onto a transparency, it can be used as described above to recursively compute weight multiplicities of irreducible modules for any of the rank 2 algebras. After the page is rotated by 180 degrees, the open circle corresponding to ρ should be placed over the weight to be computed. It will be the alternating sum of the weights under the other circles, where the plus or minus signs inscribed in the circles indicate whether to add or subtract. It is well known that for type A_2 the resulting pattern of multiplicities is easy to describe. The weight diagrams for type A_2 consist of concentric hexagonal shells, which may degenerate into triangles towards the center. The outer shell consists of weights all of whose multiplicities are equal to 1. The weights on the next hexagonal shell inward have multiplicity 2, and each successive shell inward has all multiplicities one more than the one outside it. This pattern continues until the hexagonal shell becomes a triangle. The multiplicity of each weight on that triangle, and on all weights further inward, is the same, one more than the multiplicity on that innermost hexagon. For example, in Figure 8, the weight diagram consists of two hexagonal shells and one triangular shell. The outer hexagonal shell has 15 weights, each with multiplicity equal to 1, the next hexagonal shell has 9 weights each with multiplicity equal to 2, and the inner triangular shell has 3 weights each with multiplicity 3. As a check on this, note that $42 = (15)(1) + (9)(2) + (3)(3)$ is then the dimension of the irreducible A_2-module in Figure 8. Using $n_1 = 3$ and $n_2 = 2$, this agrees with the formula

$$\dim(V^\lambda) = (n_1 + n_2 + 2)(n_1 + 1)(n_2 + 1)/2$$

for an irreducible A_2-module V^λ with $\lambda = n_1\lambda_1 + n_2\lambda_2$.

In my thesis [**F1, F2**] I studied certain patterns which occur in the tensor product decomposition of a fixed irreducible **g**-module, V^λ, with all other modules V^μ. For fixed λ, as μ varies there are only a finite number of different patterns of outer multiplicities which can occur, and there are sets of values for μ for which the pattern is constant. I called those zones of stability for tensor product decompositions, and they can be understood from the geometrical point of view of the Racah-Speiser algorithm. If the weight diagram of V^λ is shifted parallel to one of the fundamental weights, say by $\mu + m\lambda_i$, there is a least value m_i such that for $m \geq m_i$, the set of shifted weights, $\Pi^\lambda + \mu + m\lambda_i + \rho$ is contained in the union of the images of the fundamental chamber under $W(i)$, the subgroup of the Weyl group generated by the simple reflections r_j, $j \neq i$. These are the chambers containing the weights $k\lambda_i$ for $k \geq 1$. If m exceeds m_i, the only $w \in W$ which may make nonzero contributions to the outer multiplicity are those from $W(i)$, and those fix

λ_i. The geometrical reflection process which generates the tensor product multiplicities is therefore the same for each $m \geq m_i$. While the highest weights of the modules occurring increase by the number of λ_i's added, their outer multiplicities stay constant. In fact, we have the following precise result from [**F2**] about when a particular weight β of V^λ, reaches the zone of stability.

THEOREM 5.1. *Let $\lambda, \mu \in P^+$ and $\beta \in \Pi^\lambda$ be such that $\beta + \mu \in P^+$. Let*

$$\beta - r_{\beta,j}\alpha_j, \cdots, \beta, \cdots \beta + q_{\beta,j}\alpha_j$$

be the α_j weight string through β. If $\langle \mu, \alpha_j \rangle \geq q_{\beta,j}$ then

$$Mult_{\lambda,\mu}^{\beta+\mu} = Mult_{\lambda,\mu+\lambda_j}^{\beta+\mu+\lambda_j}.$$

Since $\langle \mu + \lambda_j, \alpha_j \rangle = \langle \mu, \alpha_j \rangle + 1$, it is clear that $\langle \mu, \alpha_j \rangle \geq q_{\beta,j}$ implies

$$Mult_{\lambda,\mu}^{\beta+\mu} = Mult_{\lambda,\mu+m\lambda_j}^{\beta+\mu+m\lambda_j} \qquad \text{for all } m \geq 1.$$

This result shows that for fixed $\lambda \in P^+$ and fixed $\beta \in \Pi^\lambda$, the tensor product multiplicities $Mult_{\lambda,\mu}^{\beta+\mu}$ have zones of stability as μ varies, and it is sufficient to study the finite number of μ such that $\langle \mu, \alpha_j \rangle \leq q_{\beta,j}$ for $1 \leq j \leq N-1$.

For example, using the weight diagram of $V^\lambda = V^{3\lambda_1+2\lambda_2}$ for A_2 shown in Figure 8, look at the weight $\beta = 2\lambda_1 + \lambda_2$. The α_1 weight string through this β goes from $-4\lambda_1 + 4\lambda_2 = \beta - 3\alpha_1$ to $4\lambda_1 = \beta + \alpha_1$, so $q_{\beta,1} = 1$. The α_2 weight string through this β goes from $4\lambda_1 - 3\lambda_2 = \beta - 2\alpha_2$ to $\lambda_1 + 3\lambda_2 = \beta + \alpha_2$ so $q_{\beta,2} = 1$. Theorem 5.1 then says that if $\mu = n_1\lambda_1 + n_2\lambda_2$ then $\langle \mu, \alpha_1 \rangle = n_1 \geq q_{\beta,1} = 1$ implies

$$Mult_{\lambda,\mu}^{\beta+\mu} = Mult_{\lambda,\mu+\lambda_1}^{\beta+\mu+\lambda_1}$$

and $\langle \mu, \alpha_2 \rangle = n_2 \geq q_{\beta,2} = 1$ implies

$$Mult_{\lambda,\mu}^{\beta+\mu} = Mult_{\lambda,\mu+\lambda_2}^{\beta+\mu+\lambda_2}.$$

In Figure 9 we can see the weight β, with multiplicity 2, shifted by $\mu + \rho = 2\lambda_1 + \lambda_2$, in position for the reflection process, which will reduce it by 1 because of the weight $r_2(\beta + \mu + \rho)$. Since $\mu = m\lambda_1$ for $m \geq 1$ satisfies the conditions of Theorem 5.1 for α_1, we have

$$1 = Mult_{\lambda,m\lambda_1}^{\beta+m\lambda_1} \qquad \text{for all } m \geq 1.$$

It is clear that as m increases, the reflection process yields the same result as $\beta+\mu+\rho$ shifts further along the line parallel to λ_1. In contrast, $\mu = \lambda_1$ does not satisfy the conditions of Theorem 5.1 for α_2 and we can see that adding λ_2 to μ means shifting the weight diagram in Figure 9 by λ_2, which leads to a different reflection process for the shifted β and a different multiplicity.

There is another important result about tensor product coefficients which played a role in my thesis. I will always be grateful to Prof. Bertram Kostant for drawing my attention to the following beautiful result of Parthasarathy, Ranga Rao and Varadarajan [**PRV**], which I have rewritten in the form I found most useful in my thesis.

THEOREM 5.2. [**PRV**] *Let $\lambda, \mu \in P^+$ and $\beta \in \Pi^\lambda$ be such that $\beta + \mu \in P^+$. Let $\ell = rank(\mathbf{g})$ and let $e_j \in \mathbf{g}$ be a root vector corresponding to the simple root α_j for $1 \leq j \leq \ell$. Then*

$$Mult_{\lambda,\mu}^{\beta+\mu} = dim\{v \in V_\beta^\lambda \mid e_j^{\langle \mu, \alpha_j \rangle + 1} v = 0, 1 \leq j \leq \ell\}.$$

6. Algorithms For Fusion Product Coefficients

Let $N^{(k)\ \nu}_{\lambda,\mu}$ denote the fusion product coefficient at level k. Then the Kac-Walton algorithm ([**Kac**], p. 288, [**Wal**]) expresses this as an alternating sum of tensor product multiplicities:

$$N^{(k)\ \nu}_{\lambda,\mu} = \sum_{w\in\widehat{W}} \epsilon(w) Mult^{w(\nu+\rho)-\rho}_{\lambda,\mu}$$

where $\widehat{W}$ is the affine Weyl group acting on the weight lattice of $\mathbf{g}$ with the action of the simple reflections of W as usual, but with

$$r_0(\beta) = r_\theta(\beta) + (k+\check{h})\theta.$$

Here r_θ is reflection with respect to the highest root θ of $\mathbf{g}$, and $\check{h}$ is the dual Coxeter number of $\mathbf{g}$. In the case when $\mathbf{g} = sl_N$, $\check{h} = N$, W is the symmetric group S_N, and $\theta = \sum \alpha_i$ is the sum of the simple roots of $\mathbf{g}$. Let $T_y(x) = x + y$ be the function which translates by vector y. Then it is easy to see that

$$T_{s\theta}\ r_\theta\ T_{-s\theta}(\beta) = r_\theta(\beta) + 2s\theta$$

which will equal $r_0(\beta)$ if $s = \frac{1}{2}(k+\check{h})$. Therefore, r_0 is reflection with respect to the shifted hyperplane perpendicular to θ, translated by $\frac{1}{2}(k+\check{h})\theta$.

Using the Racah-Speiser formula in the Kac-Walton formula gives a formula for fusion coefficients as an alternating sum of inner multiplicities:

$$N^{(k)\ \nu}_{\lambda,\mu} = \sum_{w\in\widehat{W}} \epsilon(w) Mult_\lambda(w(\nu+\rho) - \mu - \rho)$$

which has a nice geometrical interpretation as before, but using the affine Weyl group $\widehat{W}$ instead of W.

The only weights ν for which $N^{(k)\ \nu}_{\lambda,\mu}$ may be nonzero are those of the form $\nu = \beta + \mu$ where $\beta \in \Pi^\lambda$. The geometrical interpretation of this formula is now as follows:

(1) Shift the weight diagram of V^λ by adding $\mu + \rho$.

(2) Use the affine Weyl group to move all shifted weights into the part of the dominant chamber bounded by the reflection wall of the affine reflection, r_0, where they accumulate as an alternating sum of inner multiplicities of V^λ, adding if the required $w \in \widehat{W}$ is even, subtracting if it is odd.

(3) The resulting pattern of numbers will be non-negative integers, zero if the shifted weight is on a reflection wall, and after shifting the pattern back by subtracting ρ, you will have the fusion product coefficients.

To get a better intuitive understanding of this algorithm, it is useful to do some rank 2 cases using the diagrams from the Appendix. To include the new affine reflection, r_0, you need to make a transparency for the reflection line corresponding to the highest root, θ, and you need to know the dual Coxeter number, $\check{h}$, where

$$\theta = \begin{cases} \lambda_1 + \lambda_2, & \text{in the } A_2 \text{ case;} \\ 2\lambda_1 & \text{in the } B_2 \text{ case;} \\ \lambda_2 & \text{in the } G_2 \text{ case.} \end{cases} \qquad \check{h} = \begin{cases} 3, & \text{in the } A_2 \text{ case;} \\ 3 & \text{in the } B_2 \text{ case;} \\ 4 & \text{in the } G_2 \text{ case.} \end{cases}$$

For $\mathbf{g}$ of type A_2, level 2, the following table gives the correspondence between the fusion algebra labels [i] used in Table 3, the triples (n_0, n_1, n_2) whose sum equals the level, and the weights $\lambda = n_1\lambda_1 + n_2\lambda_2 \in P^+_2$:

Table 6: Label-Weight Correspondence for A_2 of level $k = 2$

[i]	(n_0, n_1, n_2)	$\lambda = n_1\lambda_1 + n_2\lambda_2$
[0]	$(2, 0, 0)$	0
[1]	$(0, 2, 0)$	$2\lambda_1$
[2]	$(0, 0, 2)$	$2\lambda_2$
[3]	$(0, 1, 1)$	$\lambda_1 + \lambda_2$
[4]	$(1, 0, 1)$	λ_2
[5]	$(1, 1, 0)$	λ_1

To check, for example, the fusion product $[3] \cdot [3] = [0] + [3]$ from Table 3, we would take the weight diagram of $V^{\lambda_1+\lambda_2}$, the adjoint representation, and shift it by $\mu + \rho = 2\lambda_1 + 2\lambda_2$, and use the affine Weyl group to move all shifted weights into the part of the dominant chamber bounded by the affine reflecting line. See Figure 10 and verify that after shifting back by ρ the surviving highest weights are 0 and $\lambda_1 + \lambda_2$, each with multiplicity 1. (Note that the tensor product multiplicity of $\lambda_1 + \lambda_2$ would have been 2, but the affine reflection line reduced it by one, and killed two other weights which were on it.)

In the next two tables, for $\mathbf{g}$ of type B_2, levels 1 and 2, respectively, we give the correspondence between the fusion algebra labels [i] used in Tables 1 and 5, respectively, the triples (n_0, n_1, n_2) whose sum equals the level, and the weights $\lambda = n_1\lambda_1 + n_2\lambda_2$:

Table 7: Label-Weight Correspondence for B_2 of level $k = 1$

[i]	(n_0, n_1, n_2)	$\lambda = n_1\lambda_1 + n_2\lambda_2$
[0]	$(1, 0, 0)$	0
[1]	$(0, 0, 1)$	λ_2
[2]	$(0, 1, 0)$	λ_1

Table 8: Label-Weight Correspondence for B_2 of level $k = 2$

[i]	(n_0, n_1, n_2)	$\lambda = n_1\lambda_1 + n_2\lambda_2$
[0]	$(2, 0, 0)$	0
[1]	$(0, 0, 2)$	$2\lambda_2$
[2]	$(1, 1, 0)$	λ_1
[3]	$(0, 1, 1)$	$\lambda_1 + \lambda_2$
[4]	$(0, 2, 0)$	$2\lambda_1$
[5]	$(1, 0, 1)$	λ_2

We may check the level 1 fusion products from Table 1 as follows. For $[1] \cdot [1] = [0]$, take the weight diagram of V^{λ_2}, shift it by $\lambda_2 + \rho = \lambda_1 + 2\lambda_2$, and use the affine Weyl group to move all shifted weights into the part of the dominant chamber bounded by the affine reflecting line. See Figure 11 and verify that after shifting back by ρ the only surviving highest weight is 0. For $[1] \cdot [2] = [2]$, shift the diagram of V^{λ_2} by $\lambda_1 + \rho = 2\lambda_1 + \lambda_2$, and after the same process (see Figure 12) find the only surviving highest weight is λ_1. For $[2] \cdot [2] = [0] + [1]$, shift the weight diagram of V^{λ_1} by $\lambda_1 + \rho = 2\lambda_1 + \lambda_2$, and after reflecting (see Figure 13) find the only surviving highest weights are 0 and λ_2.

In Figure 14 check the level 2 fusion product $[3] \cdot [3] = [0] + [4] + [5]$ from Table 5 by shifting the weight diagram of $V^{\lambda_1+\lambda_2}$ by $\lambda_1 + \lambda_2 + \rho = 2\lambda_1 + 2\lambda_2$, and after the affine reflection process (with the affine reflection line located as it should be for level 2) verify that (after shifting back by ρ) the surviving highest weights are 0, $2\lambda_1$ and λ_2.

In comparing the Kac-Walton algorithm with the one for $Mult^{\nu}_{\lambda,\mu}$, we see that the shifting is the same, and all reflections coming from $w \in W$ are the same, but there are more contributions from the extra elements in $\widehat{W}$. Elements of W are sufficient to reflect all weights of the diagram into the dominant chamber, but some may be on the side of the reflection wall of r_0 not containing the origin. One application of r_0 would then move the weight to the other side, but perhaps take it out of the dominant chamber, requiring more reflections from W to move it back into the dominant chamber. For example, in Figure 15, which is just Figure 9 with the affine reflection line added in the position appropriate for level 5, we see that the shifted highest weight falls on the affine reflection line. So the tensor product decomposition given in the last section is truncated by removing that highest weight on the line to give the corresponding fusion product. This can be seen more clearly in Figure 16, where the smaller module V^{λ_1} is shifted by $3\lambda_1 + 2\lambda_2 + \rho$. In Figure 17 we see an example where many weights of the shifted module are on the far side of the affine reflection line, and where application of r_0 does not bring the weight into P^+_k.

We would like to briefly discuss how the result in Theorem 5.1 on zones of stability for tensor product multiplicities might give some information about such zones for fusion coefficients. It is clear that $P^+_k \subseteq P^+_{k+1}$, and for fixed λ and μ, increasing level k means that the affine reflection wall will move further away from the origin. For α any root, the reflection with respect to the hyperplane perpendicular to α is

$$r_\alpha(\lambda) = \lambda - \langle \lambda, \alpha \rangle \ \alpha$$

and if we write $T_\beta(\lambda) = \lambda + \beta$ for translation by β, then

$$r_0 r_\theta(\lambda) = \lambda + (k + \check{h})\theta = T_{(k+\check{h})\theta}(\lambda)$$

is a translation in $\widehat{W}$. It is easy to check the relation $r_\alpha T_\beta r_\alpha = T_{r_\alpha(\beta)}$, which implies that

$$w T_{(k+\check{h})\theta} w^{-1} = T_{(k+\check{h})w(\theta)}$$

is a translation in $\widehat{W}$ for each $w \in W$. Since θ is the highest root of $\mathbf{g}$, it's orbit under W is the set of all long roots. So for each long root, β, the translation $T_{(k+\check{h})\beta}$ and its inverse are even elements of $\widehat{W}$. These translations generate an abelian subgroup $\mathcal{T}$ of $\widehat{W}$ and the relation $wT_\beta w^{-1} = T_{w(\beta)}$ shows that $\widehat{W}$ is the semi-direct product of W and $\mathcal{T}$. A fundamental domain for the action of $\widehat{W}$ on the set of all weights P can be determined by writing any element of $\widehat{W}$ as a translation from $\mathcal{T}$ followed by an element of W. The translations $T_{(k+\check{h})n\beta}$, $n \in \mathbf{Z}$, allow any weight λ to be moved to a weight μ such that $-(k + \check{h}) \leq \langle \mu, \beta \rangle \leq (k + \check{h})$. These inequalities say that μ is between the shifted hyperplanes fixed by $T_{(k+\check{h})\beta} r_\beta$ and by $r_\beta T_{(k+\check{h})\beta}$. Doing this for each positive long root β allows us to move λ to a weight μ in the closure of a fundamental domain for $\mathcal{T}$,

$$\mathcal{F}_k = \{\mu \in P \mid -(k + \check{h}) \leq \langle \mu, w\theta \rangle \leq (k + \check{h}), \forall w \in W\},$$

the region bounded by all such pairs of shifted hyperplanes. That region is obviously W-invariant, and each weight in it can be moved by W into the dominant chamber, P^+. So a fundamental domain for $\widehat{W}$ would be the intersection $\mathcal{F}_k \cap P^+ = P^+_{k+\check{h}}$ and

$$\mathcal{F}_k = W(P^+_{k+\check{h}}) = \bigcup_{w\in W} w(P^+_{k+\check{h}}).$$

Let

$$\mathcal{F}'_k = \{\mu \in P \mid -(k+\check{h}) < \langle \mu, w\theta\rangle < (k+\check{h}), \forall w \in W\}$$

be the interior of $\mathcal{F}_k$. Then $\mathcal{F}'_k$ is also W-invariant, and for any translation $T \in \mathcal{T}$, if $\mathcal{F}'_k \cap T(\mathcal{F}'_k)$ is nonempty then T is the identity element. The boundary walls of $\mathcal{F}_k$ expand as k increases so there is a minimum value of k for which $\Pi^\lambda + \mu + \rho \subseteq \mathcal{F}'_k$. For example, in Figure 9 we see that the shifted weight diagram is contained in the interior of the large hexagon, $\mathcal{F}'_6$.

THEOREM 6.1. *For $\lambda, \mu \in P^+$, if k is large enough so that $\Pi^\lambda + \mu + \rho \subseteq \mathcal{F}'_k$, then for all $\nu \in P^+_k$ we have*

$$N^{(k)\ \nu}_{\lambda,\mu} = Mult^\nu_{\lambda,\mu}.$$

Proof: For $\lambda, \mu \in P^+$ we see that if $\Pi^\lambda + \mu + \rho \subseteq \mathcal{F}'_k$ then only elements of W can bring those shifted weights into $P^+_{k+\check{h}}$, and none go on the fixed hyperplane of r_0. When using the Kac-Walton algorithm to compute the fusion coefficients in the product $[\lambda]\cdot[\mu]$ this condition guarantees that the only nonzero contributions may come from affine Weyl group elements which are actually in W, matching the expression in the Racah-Speiser algorithm for the tensor product coefficients and giving the equality of the fusion and tensor product coefficients as claimed. □

I have the following result for finding that minimum value of k which makes the above happen.

THEOREM 6.2. *For $\lambda, \mu \in P^+$, we have*

$$\langle \lambda+\mu, \theta\rangle \le k \text{ if and only if } \Pi^\lambda + \mu + \rho \subseteq \mathcal{F}'_k.$$

Proof: First note that $\langle \lambda+\mu, \theta\rangle \le k$ is equivalent to $\langle \lambda+\mu+\rho, \theta\rangle < k + \check{h}$ because $\langle \lambda+\mu+\rho, \theta\rangle = (\lambda+\mu+\rho, \theta) = (\lambda+\mu, \theta) + (\rho, \theta) = \langle \lambda+\mu, \theta\rangle + \check{h} - 1$. Since $\lambda \in \Pi^\lambda$, if $\Pi^\lambda + \mu + \rho \subseteq \mathcal{F}'_k$ then $\lambda+\mu+\rho \in \mathcal{F}'_k$, so $\langle \lambda+\mu+\rho, \theta\rangle < k+\check{h}$, so we get $\langle \lambda+\mu, \theta\rangle \le k$.

Now suppose that we have the above inequality. To show the containment we break the argument into two steps. We will show

$$(1)\quad \Pi^{\lambda+\mu+\rho} \subseteq \mathcal{F}'_k \qquad \text{and then we will show} \qquad (2)\quad \Pi^\lambda + \mu + \rho \subseteq \Pi^{\lambda+\mu+\rho}.$$

Both $\Pi^{\lambda+\mu+\rho}$ and $\mathcal{F}'_k$ are W-invariant sets, so each consists of the W-conjugates of their dominant integral elements. So if $\Pi^{\lambda+\mu+\rho} \cap P^+ \subseteq \mathcal{F}'_k \cap P^+$ then we get (1). For any $\beta \in \Pi^{\lambda+\mu+\rho} \cap P^+$, we know $\beta \le \lambda+\mu+\rho$, so $\beta = \lambda+\mu+\rho - \sum_{i=1}^{N-1} k_i\alpha_i$ with $0 \le k_i \in \mathbf{Z}$. Then we have

$$\langle \beta, \theta\rangle = \langle \lambda+\mu+\rho, \theta\rangle - \sum_{i=1}^{N-1} k_i \langle \alpha_i, \theta\rangle.$$

We also know that $\langle\alpha_i,\theta\rangle = (\alpha_i,\theta) = (\theta,\alpha_i) = \langle\theta,\alpha_i\rangle\ (\alpha_i,\alpha_i)/2$, but $(\alpha_i,\alpha_i) > 0$ and $\theta \in P^+$ since it is the highest weight of the adjoint representation, so $\langle\theta,\alpha_i\rangle \geq 0$. Therefore,

$$\langle\beta,\theta\rangle \leq \langle\lambda+\mu+\rho,\theta\rangle < k+\check{h}$$

so $\beta \in \mathcal{F}'_k \cap P^+$.

It is well-known that for any $\lambda \in P^+$,

$$\Pi^\lambda = \{w\beta \mid \beta \in P^+, \beta \leq \lambda, w \in W\}.$$

Let $\beta \in \Pi^\lambda$ so $\beta \leq \lambda$ and $\beta+\mu+\rho \leq \lambda+\mu+\rho$. For some $w \in W$ we have $w(\beta+\mu+\rho) \in P^+$. We know $\mu \geq w\mu$ and $\rho \geq w\rho$ since $\mu,\rho \in P^+$, and $\lambda \geq w\beta$ since $w\beta \in \Pi^\lambda$. Then

$$\lambda+\mu+\rho - w(\beta+\mu+\rho) = (\lambda - w\beta) + (\mu - w\mu) + (\rho - w\rho) \geq 0.$$

But $\lambda+\mu+\rho \geq w(\beta+\mu+\rho) \in P^+$ means $w(\beta+\mu+\rho) \in \Pi^{\lambda+\mu+\rho}$ so $\beta+\mu+\rho \in \Pi^{\lambda+\mu+\rho}$ by the W-invariance of $\Pi^{\lambda+\mu+\rho}$. Note that this proof of (2) does not use the inequality involving k. $\square$

In [**FZ**] the following formula for fusion coefficients for affine algebras was obtained using the theory of vertex operator algebras. (Also see [**GW**].)

THEOREM 6.3. [**FZ**] *Let $\lambda,\mu,\nu \in P_k^+$, and let e_θ be a root vector of $\mathbf{g}$ in the θ root space of $\mathbf{g}$. Let $v_\lambda^\lambda \in V^\lambda$ be a highest weight vector. Then the level k fusion coefficient $N_{\lambda,\mu,\nu}^{(k)}$ equals the dimension of the vector space $T_k(\lambda,\mu,\nu) =$*

$$\{f \in Hom_{\mathbf{g}}(V^\lambda \otimes V^\mu \otimes V^\nu, \mathbf{C}) \mid f(v_\lambda^\lambda \otimes e_\theta^{k-\langle\lambda,\theta\rangle+1} v^\mu \otimes v^\nu) = 0,\ \forall v^\mu \in V^\mu,\ \forall v^\nu \in V^\nu\}.$$

It is clear that the k-dependent condition on f in $T_k(\lambda,\mu,\nu)$ will be trivially satisfied for any ν when the operator $e_\theta^{k-\langle\lambda,\theta\rangle+1}$ is the zero operator on V^μ, and in that case $dim(T_k(\lambda,\mu,\nu)) = Mult_{\lambda,\mu,\nu}^0 = Mult_{\lambda,\mu}^{\nu^+}$ equals the multiplicity of the trivial module in the triple tensor product $V^\lambda \otimes V^\mu \otimes V^\nu$ which equals the multiplicity of the contragrediant module V^{ν^+} in $V^\lambda \otimes V^\mu$. Consider the decomposition of V^μ into irreducible sl_2-modules with respect to the subalgebra $\mathbf{g}_\theta \subseteq \mathbf{g}$ with basis e_θ, f_θ in the $-\theta$ root space of $\mathbf{g}$ and $h_\theta = [e_\theta, f_\theta]$. It is well-known that any finite dimensional irreducible representation $V(n)$ of sl_2 is uniquely determined by it's highest eigenvalue for h_θ, $0 \leq n \in \mathbf{Z}$, and that $dim(V(n)) = n+1$. Using the well-known action of e_θ on $V(n)$, it is easy to see that e_θ^{n+1} is the zero operator on $V(n)$. In the decomposition of V^μ into sl_2-modules, there is a component $V(n)$ with largest n, and so e_θ^{n+1} is the zero operator on that and all other components. It is not hard to see that the largest n is $\langle\mu,\theta\rangle$, which corresponds to the $\mathbf{g}_\theta$-submodule generated by the highest weight vector of V^μ. Then the combined results of Theorems 6.1 and 6.2 follow from Theorem 6.3 because the condition on k which guarantees equality of fusion and tensor coefficients is that

$$k - \langle\lambda,\theta\rangle + 1 \geq \langle\mu,\theta\rangle + 1.$$

It is interesting to see how the geometrical aspects of the Kac-Walton and Racah-Speiser algorithms give this same result.

If we do not demand equality of fusion and tensor product coefficients for all weights of the shifted weight diagram, we can still get a condition which guarantees it for a fixed weight of Π^λ.

For each $\beta \in \Pi^\lambda$ such that $\beta + \mu \in P^+$, there is a minimum value of k, denoted by $k_{max} = k_{max}(\beta, \lambda, \mu)$, such that for any $w \in \widehat{W}$, $w(\beta + \mu + \rho) \in \Pi^\lambda + \mu + \rho$ implies $w \in W$. Assuming that $k \geq \langle \lambda, \theta \rangle$ and $k \geq \langle \mu, \theta \rangle$ so that $\lambda, \mu \in P_k^+$, and that $k \geq \langle \beta + \mu, \theta \rangle$ so that $\beta + \mu \in P_k^+$, if $k \geq k_{max}$ then the discussion above shows that

$$N_{\lambda,\mu}^{(k)\ (\beta+\mu)} = Mult_{\lambda,\mu}^{\beta+\mu}.$$

CONJECTURE 6.4. *For $\lambda, \mu \in P^+$, $\beta \in \Pi^\lambda$ and k large enough so that $\lambda, \mu, \beta + \mu \in P_k^+$, suppose that $r_0(\beta + \mu + \rho) \notin \Pi^\lambda + \mu + \rho$. Then for any $w \in \widehat{W}$, we have*

$$w(\beta + \mu + \rho) \in \Pi^\lambda + \mu + \rho \qquad \textit{implies} \qquad w \in W.$$

I re-discovered the following conjecture, which appeared in [**Wal2**] without proof. (Thanks to Mark Walton for informing me of his paper after seeing an earlier version of this paper on the internet arXiv.) As far as I know, it remains unproven, but will be the subject of a subsequent publication if I can prove it.

CONJECTURE 6.5. *For $\lambda, \mu \in P_k^+$, $\beta \in \Pi^\lambda$ such that $\beta + \mu \in P_k^+$, we have $N_{\lambda,\mu}^{(k)\ (\beta+\mu)}$ equals the dimension of the space*

$$F_k^+(\lambda, \beta, \mu) = \{v \in V_\beta^\lambda \mid e_j^{\langle \mu, \alpha_j \rangle + 1} v = 0, 1 \leq j \leq \ell, \ \textit{and}\ e_\theta^{k - \langle \beta + \mu, \theta \rangle + 1} v = 0\}.$$

This conjecture is a blending of the PRV and FZ theorems, showing that the FZ theorem is actually a beautiful generalization of the PRV theorem. It implies the following result, which tells the level k at which the fusion coefficient associated with a single weight $\beta \in \Pi^\lambda$ equals the tensor product multiplicity associated with that weight.

COROLLARY 6.6. *Suppose $\lambda, \mu \in P_k^+$, and $\beta \in \Pi^\lambda$ is such that $\beta + \mu \in P_k^+$. Let the θ weight string through β in Π^λ be $\beta - r\theta, \cdots, \beta, \cdots, \beta + q\theta$. Then $k \geq \langle \mu, \theta \rangle + r$ implies $N_{\lambda,\mu}^{(k)\ (\beta+\mu)} = Mult_{\lambda,\mu}^{\beta+\mu}$.*

Proof: If the condition $e_\theta^{k - \langle \beta + \mu, \theta \rangle + 1} v = 0$ is satisfied for all $v \in V_\beta^\lambda$ then $F_k^+(\lambda, \beta, \mu) = V^+(\lambda, \beta, \mu)$ whose dimension is the tensor product multiplicity $Mult_{\lambda,\mu}^{\beta+\mu}$. But that condition will be satisfied when $k - \langle \beta + \mu, \theta \rangle + 1 > q$ because that many applications of the operator e_θ will move v just beyond the $\beta + q\theta$ weight space in the string. We know that $r = r_{\lambda,\beta}$ and $q = q_{\lambda,\beta}$ depend on λ and on β, and satisfy $r - q = \langle \beta, \theta \rangle$, so the inequality above is equivalent to $k \geq \langle \mu, \theta \rangle + r$.

In this approach to fusion coefficients, for fixed values of λ, μ and ν, as the level k varies, we try to determine for what level k_{max} they reach their maximum, the tensor product coefficient. This should be compared to the use of "threshold levels" in [**BKMW**]. The spaces $T_k(\lambda, \mu, \nu)$ for fixed weights as k increases form a filtration of the largest such space, which is when $k \geq k_{max}(\beta, \lambda, \mu)$, $\nu^+ = \beta + \mu$. There is also a $k_{min} = k_{min}(\beta, \lambda, \mu)$ such that $N_{\lambda,\mu}^{(k)\ (\beta+\mu)} = 0$ for $k < k_{min}$ but $N_{\lambda,\mu}^{(k)\ (\beta+\mu)} \neq 0$ for $k = k_{min}$. We may choose a basis B_k of each space T_k, $k_{min} \leq k \leq k_{max}$ so that each B_k is an extension of B_{k-1}. Then for each basis vector, $v \in B_{k_{max}}$ there is a smallest k_v such that the vector is in B_{k_v}, and that k_v is called the threshold level of v. Knowing the list of all threshold levels is equivalent to knowing all the fusion coefficients as k varies, but since there is no canonical choice of basis in the spaces T_k, it seems more natural to focus on the dimensions of the spaces T_k.

Finally, the tensor product multiplicity $Mult^{\nu}_{\lambda,\mu}$ may be part of a zone of uniform decomposition, and equal to another one with μ and ν reduced. For example, in Figure 18 we see the weight diagram from Figure 15 shifted by an additional λ_1, which changes the W-reflection process for many of the weights, but not for the weight $2\lambda_1 + \lambda_2$ which is in its zone of stability along the λ_1 line. But when doing the Kac-Walton algorithm, the affine reflection cancels that tensor product coefficient because of its r_0 symmetry with the shifted highest weight. But if the level k is increased, then that symmetry is broken and that fusion coefficient remains constant for all $k \geq 6$.

7. A Different Approach

Let $G = \mathbf{Z}_N^k$. The symmetric group S_k acts on G by permuting the k-tuples. For $a \in G$, let $[a]$ be the orbit of a and $\mathcal{O}$ be the set of all orbits. These orbits are precisely the subsets

$$P(i_0, i_1, \ldots, i_{N-1}) = \{x \in \mathbf{Z}_N^k \mid j \text{ occurs exactly } i_j \text{ times in } x, 0 \leq j \leq N-1\}$$

where $(i_0, i_1, \cdots, i_{N-1})$ is any N-tuple of nonnegative integers such that

$$i_0 + i_1 + \cdots + i_{N-1} = k.$$

We now have a bijection between $\mathcal{O}$ and the set P_k^+ when $\mathbf{g}$ is of type A_{N-1}. For $[a], [b], [c] \in \mathcal{O}$ we believe the fusion coefficients $N_{[a],[b]}^{[c]}$ have a combinatorial description in terms of the group G. The conjugate of $[c]$ is $[-c]$ and we prefer to study the totally symmetric coefficients

$$N_{[a],[b],[c]} = N_{[a],[b]}^{[-c]}.$$

We consider the following combinatorial question. For $[a], [b], [c] \in \mathcal{O}$, the group S_k acts on

$$T([a],[b],[c]) = \{(x, y, z) \in [a] \times [b] \times [c] \mid x + y + z = 0\}$$

which decomposes into a finite number of orbits under that action. Let the number of such orbits be denoted by $M([a],[b],[c])$. Determine $M([a],[b],[c])$ and show how it is related to $N_{[a],[b],[c]}$. For $N = 2, 3$ we have the following results.

THEOREM 7.1. *For $N = 2$, for any integral level $k \geq 1$, with notation as above, we have*

$$M([a],[b],[c]) = N_{[a],[b],[c]}.$$

THEOREM 7.2. *For $N = 3$, for any integral level $k \geq 1$, with notation as above, we have*

$$M([a],[b],[c]) = \binom{N_{[a],[b],[c]} + 1}{2}.$$

In previous work with F. Akman [**AFW**], we introduced the idea of covering a fusion algebra by a finite abelian group and proved that the (p, q)-minimal model fusion algebra, which comes from the discrete series of $0 < c < 1$ representations of the Virasoro algebra, can be covered by the group Z_2^{p+q-5}. The basic idea, which is only set up to handle fusion algebras whose fusion coefficients N_{ij}^k are in $\{0, 1\}$, is as follows.

Definition. Let $(G,+,0)$ be a finite abelian group and let $G = P_0 \cup P_1 \cup ... \cup P_{N-1}$ be a partition into N disjoint subsets with $P_0 = \{0\}$. Let W be an N-dimensional vector space over $\mathbf{Q}$ with basis $P = \{P_0, P_1, ..., P_{N-1}\}$ and define a bilinear multiplication on W by the formula

$$P_i * P_j = \sum_{k \in T(i,j)} P_k$$

where

$$T(i,j) = \{k \mid \exists a \in P_i, \exists b \in P_j,\ a+b \in P_k\}.$$

We say that such a partition is associative if the product $*$ is associative. We say that a group G covers a fusion algebra F if there is an associative partition P of G and a bijection Φ between A and P which gives an algebra isomorphism between F and W such that $\Phi(\Omega) = P_0$.

As an example of a nontrivial fusion algebra which can be covered by a group, let the $\mathcal{W}$ algebra coming from the coset construction of

$$\frac{SU(N)_r \otimes SU(N)_s}{SU(N)_{r+s}}$$

be denoted by $\mathcal{W}_N(r,s)$. The fusion rules for $\mathcal{W}_3(1,1)$ are as follows.

Table 9: Fusion rules for $\mathcal{W}_3(1,1)$

[a]×[b]	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]		[0]+[1]	[3]	[2]+[3]	[5]	[4]+[5]
[2]			[4]	[5]	[0]	[1]
[3]				[4]+[5]	[1]	[0]+[1]
[4]					[2]	[3]
[5]						[2]+[3]

Note that $\{[0],[2],[4]\}$ forms a subgroup isomorphic to $\mathbf{Z}_3$. We find that $\mathbf{Z}_3^2$ covers these fusion rules as follows:

$$\{(0,0)\} \leftrightarrow [0] \qquad \{(1,2),(2,1)\} \leftrightarrow [1] \qquad \{(1,1)\} \leftrightarrow [2]$$
$$\{(0,2),(2,0)\} \leftrightarrow [3] \qquad \{(2,2)\} \leftrightarrow [4] \qquad \{(1,0),(0,1)\} \leftrightarrow [5]$$

Let $\mathbf{g} = sl_2$. There are $k+1$ $\hat{g}$-modules $\hat{V}^a$ of level $k \geq 1$, indexed by spin $a \in \frac{1}{2}\mathbf{Z}$ with $0 \leq a \leq \frac{k}{2}$. We have the tensor product decomposition

$$V^a \otimes V^b = \sum_{|a-b| \leq c \leq a+b} V^c$$

where the sum is only taken over those $c \in \frac{1}{2}\mathbf{Z}$ such that $a+b+c \in \mathbf{Z}$. The fusion rules for level k are a simple truncation of that summation:

$$N_{a,b}^c = \begin{cases} 1, & \text{if } |a-b| \leq c \leq a+b,\ a+b+c \in \mathbf{Z},\ a+b+c \leq k; \\ 0, & \text{otherwise.} \end{cases}$$

The conditions above imply that $c \leq \frac{k}{2}$.

Alternative way: Re-index the modules $\hat{V}^a$ on level k by $m = 2a + 1 \in \mathbf{Z}$ with $1 \leq m \leq k + 1$. Then $m = dim(V^a)$ and we write $\hat{V}^a = \hat{V}(2a + 1) = \hat{V}(m)$. Let $p = k + 2$.

Definition. For integer $p \geq 2$ the triple of integers (m, m', m'') is **p-admissible** when $0 < m, m', m'' < p$, the sum $m + m' + m'' < 2p$ is odd, and the "triangle" inequalities

$$m < m' + m'', \quad m' < m + m'', \quad m'' < m + m'$$

are satisfied.

Then the level k sl_2 fusion rules are: $N_{m,m'}^{m''} = 1$ if (m, m', m'') is p-admissible, $N_{m,m'}^{m''} = 0$ otherwise.

THEOREM 7.3. *The level k fusion rules for $\mathbf{g} = sl_2$ define a fusion algebra F with*

$$A = \{m \in \mathbf{Z} \mid 1 \leq m \leq k + 1\},$$

distinguished element $\Omega = 1$ and the conjugate $m^+ = m$. F is covered by the elementary abelian 2-group $G = \mathbf{Z}_2^k$ with partition given by

$$P_i = \{g \in G \mid \textit{exactly } i \textit{ coordinates of } g \textit{ are } 1\}$$

for $0 \leq i \leq k$.

The following tables illustrate how the fusion tables for A_1 on levels 2 and 3 are covered.

Table 10: Group $\mathbf{Z}_2^2$ covering the Fusion Table for A_1 of level $k = 2$

[i]·[j]	(0,0)	(1,1)	(1,0),(0,1)
(0,0)	(0,0)	(1,1)	(1,0),(0,1)
(1,1)		(0,0)	(0,1),(1,0)
(1,0),(0,1)			(0,0),(1,1) (1,1),(0,0)

Table 11: Group $\mathbf{Z}_2^3$ covering the Fusion Table for A_1 of level $k = 3$

[i]×[j]	(0,0,0)	(1,1,1)	(0,1,1),(1,0,1),(1,1,0)	(1,0,0),(0,1,0),(0,0,1)
(0,0,0)	(0,0,0)	(1,1,1)	(0,1,1),(1,0,1),(1,1,0)	(1,0,0),(0,1,0),(0,0,1)
(1,1,1)		(0,0,0)	(1,0,0),(0,1,0),(0,0,1)	(0,1,1),(1,0,1),(1,1,0)
(0,1,1) (1,0,1) (1,1,0)			(0,0,0),(1,1,0),(1,0,1) (1,1,0),(0,0,0),(0,1,1) (1,0,1),(0,1,1),(0,0,0)	(1,1,1),(0,0,1),(0,1,0) (0,0,1),(1,1,1),(1,0,0) (1,0,0),(1,0,0),(1,1,1)
(1,0,0) (0,1,0) (0,0,1)				(0,0,0),(1,1,0),(1,0,1) (1,1,0),(0,0,0),(0,1,1) (1,0,1),(0,1,1),(0,0,0)

The following table illustrates how the fusion table for A_2 of level 2 can be covered. In this case all fusion coefficients are 0 or 1, but in order to make this idea work for higher levels of A_2, where the fusion coefficients can be greater than 1, the method of Theorem 7.2 must be used.

Table 12: Group $\mathbf{Z}_3^2$ covering the Fusion Table for A_2 of level $k=2$

[a]×[b]	(0,0)	(1,1)	(2,2)	$(1,2)$ $(2,1)$	$(2,0)$ $(0,2)$	$(1,0)$ $(0,1)$
(0,0)	(0,0)	(1,1)	(2,2)	$(1,2)$ $(2,1)$	$(2,0)$ $(0,2)$	$(1,0)$ $(0,1)$
(1,1)		(2,2)	(0,0)	$(2,0)$ $(0,2)$	$(0,1)$ $(1,0)$	$(2,1)$ $(1,2)$
(2,2)			(1,1)	$(0,1)$ $(1,0)$	$(1,2)$ $(2,1)$	$(0,2)$ $(2,0)$
$(1,2)$ $(2,1)$				$(2,1),(0,0)$ $(0,0),(1,2)$	$(0,2),(1,1)$ $(1,1),(2,0)$	$(2,2),(0,1)$ $(1,0),(2,2)$
$(2,0)$ $(0,2)$					$(1,0),(2,2)$ $(2,2),(0,1)$	$(0,0),(1,2)$ $(2,1),(0,0)$
$(1,0)$ $(0,1)$						$(2,0),(1,1)$ $(1,1),(0,2)$

I would like to conclude this section with a discussion of the fusion table for A_2 of level 3, and explain how Table 4 comes from Theorem 7.2. We must look at the orbits of $\mathbf{Z}_3^3$ under the symmetric group S_3. In the notation used by Schellekens, the primaries $[i]$ for $0 \le i \le 9$ correspond to triples (i_2, i_1, i_0) with sum $i_0+i_1+i_2=3$, as follows:

$$
\begin{array}{lllll}
[0] \leftrightarrow (0,0,3) & [1] \leftrightarrow (0,3,0) & [2] \leftrightarrow (3,0,0) & [3] \leftrightarrow (0,1,2) & [4] \leftrightarrow (1,2,0) \\
[5] \leftrightarrow (2,0,1) & [6] \leftrightarrow (0,2,1) & [7] \leftrightarrow (2,1,0) & [8] \leftrightarrow (1,0,2) & [9] \leftrightarrow (1,1,1)
\end{array}
$$

Each triple (i_2, i_1, i_0) corresponds to an orbit in $\mathbf{Z}_3^3$ consisting of those triples with i_2 2's, i_1 1's and i_0 0's. In particular, for the primaries given in the earlier partial table, we have

$$
\begin{aligned}
[0] \leftrightarrow (0,0,3) &\leftrightarrow \text{ orbit of } (0,0,0) \in \mathbf{Z}_3^3 \\
[1] \leftrightarrow (0,3,0) &\leftrightarrow \text{ orbit of } (1,1,1) \in \mathbf{Z}_3^3 \\
[2] \leftrightarrow (3,0,0) &\leftrightarrow \text{ orbit of } (2,2,2) \in \mathbf{Z}_3^3 \\
[9] \leftrightarrow (1,1,1) &\leftrightarrow \text{ orbit of } (0,1,2) \in \mathbf{Z}_3^3
\end{aligned}
$$

Denoting the S_3-orbit of $(i_2, i_1, i_0) \in \mathbf{Z}_3^3$ by $[(i_2, i_1, i_0)]$, we see that each of the orbits

$$[(0,0,0)] = \{(0,0,0)\}, \quad [(1,1,1)] = \{(1,1,1)\}, \quad [(2,2,2)] = \{(2,2,2)\}$$

consists of only one element of $\mathbf{Z}_3^3$, but the orbit

$$[(0,1,2)] = \{(0,1,2),(0,2,1),(1,0,2),(1,2,0),(2,0,1),(2,1,0)\}$$

consists of the six distinct elements. Looking at the set $T([a],[b],[c])$ when $[a]$, $[b]$ and $[c]$ are singleton orbits chosen from among $[0]$, $[1]$ and $[2]$, is the same as looking at just one equation,

$$(i,i,i)+(j,j,j)+(k,k,k)=(0,0,0)$$

which has one solution when $i+j+k=0$ in $\mathbf{Z}_3^3$, none otherwise. This corresponds to the $\mathbf{Z}_3$ subtable generated just by $[0]$, $[1]$ and $[2]$. To understand the rest of the table, it might be easier to look at the S_3 orbits of equations $x+y=z$ for $x \in [a]$, $x \in [b]$ and $z \in [c]$. When $[a] = [(i,i,i)]$ is a singleton and $[b] = [9] = [(0,1,2)]$, we are looking at equations of the form

$$(i,i,i)+(r,s,t)=(r',s',t')$$

where r,s,t are all distinct, which gives $r'=r+i$, $s'=s+i$ and $t'=t+i$ are also all distinct, so $[(r',s',t')] = [9]$. There is only one S_3-orbit of such equations for fixed $[a]$ and $[b]$ so Theorem 7.2 says that $1 = M([a],[9],[9]) = N_{[a],[9],[9]}(N_{[a],[9],[9]}+1)/2$ so $N_{[a],[9],[9]} = 1$, and $0 = M([a],[9],[c])$ for $[c] \neq [9]$. Finally, to look at $[9]\cdot[9]$ we must look at equations of the form

$$(r,s,t)+(r',s',t')=(i,j,k)$$

where r,s,t are all distinct and r',s',t' are all distinct. The 36 possibilities for (i,j,k) include $(0,0,0)$ six times, $(1,1,1)$ six times, $(2,2,2)$ six times, and each of the six elements of $[9]$ occurs 3 times. One finds that each group of six with $i=j=k$ is a single S_3-orbit, but the 18 equations with i,j,k distinct fall into three orbits, the diagonal orbit of $(0,1,2)+(0,1,2)=(0,2,1)$, the orbit of $(0,1,2)+(1,2,0)=(1,0,2)$, and the orbit of $(1,2,0)+(0,1,2)=(1,0,2)$. Theorem 7.2 then says $1 = M([9],[9],[a]) = N_{[9],[9],[a]}(N_{[9],[9],[a]}+1)/2$ so $N_{[9],[9],[a]} = 1$ for $[a] = [0],[1],[2]$, and $3 = M([9],[9],[9]) = N_{[9],[9],[9]}(N_{[9],[9],[9]}+1)/2$ so $N_{[9],[9],[9]} = 2$.

8. A Connection With Ramanujan

While studying the orbits of S_k acting on $\mathbf{Z}_n^k$ (in collaboration with Michael Weiner and Matthias Beck) we noticed the following. Let

$$A(n,k,r) = \{(a_1,\cdots,a_k) \in \mathbf{Z}_n^k \mid a_1+\cdots+a_k \equiv r \mod n\}$$

and let $M(n,k,r)$ be the number of orbits of $A(n,k,r)$ under the action of S_k. Equivalently, we can represent each such orbit uniquely by a k–tuple of integers $(a_1,\cdots,a_k)$ where $0 \leq a_j \leq n-1$ for $0 \leq j \leq k$, and

$$a_1 \geq a_2 \geq \cdots \geq a_k.$$

Each such k–tuple corresponds to a partition of $a_1+\cdots+a_k$ into at most k parts, each of which is at most $n-1$. Hence, if we denote by $p(a,b,t)$ the number of partitions of t into at most b parts, each of which is at most a, we get the alternative description

$$M(n,k,r) = \sum_{t\geq 0} p(n-1,k,r+nt).$$

(Here we understand that $p(a, b, 0) = 1$ and that $0 \le r < n$.) We started to study sums of this type, and proved the beautiful formula

$$M(n,k,r) = \frac{1}{n+k} \sum_{d|g} \binom{\frac{n+k}{d}}{\frac{n}{d}} c_d(r),$$

where $g = \gcd(n, k)$ and the sum is over the positive divisors of g. Here $c_d(r)$ denotes the **Ramanujan sum**, defined for integers d and r, $d > 0$, as

$$c_d(r) = \sum_{\substack{m=0 \\ \gcd(m,d)=1}}^{d-1} \zeta_d^{mr},$$

where $\zeta_d = e^{2\pi \mathbf{i}/d}$. One immediately gets the symmetry

$$M(n, k, r) = M(k, n, r).$$

But after posting our results on the internet archives, we learned that this result was already published in 1999 by Elashvili, Jibladze and Pataraia [**EJP**]. Further research in the literature led me back to a 1902 paper by von Sterneck [**vonS**], who studied partitions into distinct parts, and to Bachmann [**Bac**] (Vol. 2, 222–241), who also obtained in 1910 a recursive formula for the number of partitions with repetitions allowed, and then to Ramanathan [**Ram**], who found in 1944 the role of Ramanujan sums in these formulas, but did not obtain the beautiful symmetry above. I mentioned this history because it shows the far reaching influence of the great mathematician, Ramanujan, whose ideas continue to affect the development of mathematics, and in whose name we meet at this International Symposium.

9. Appendices

Figure 1: Weight Lattice of Type A_2

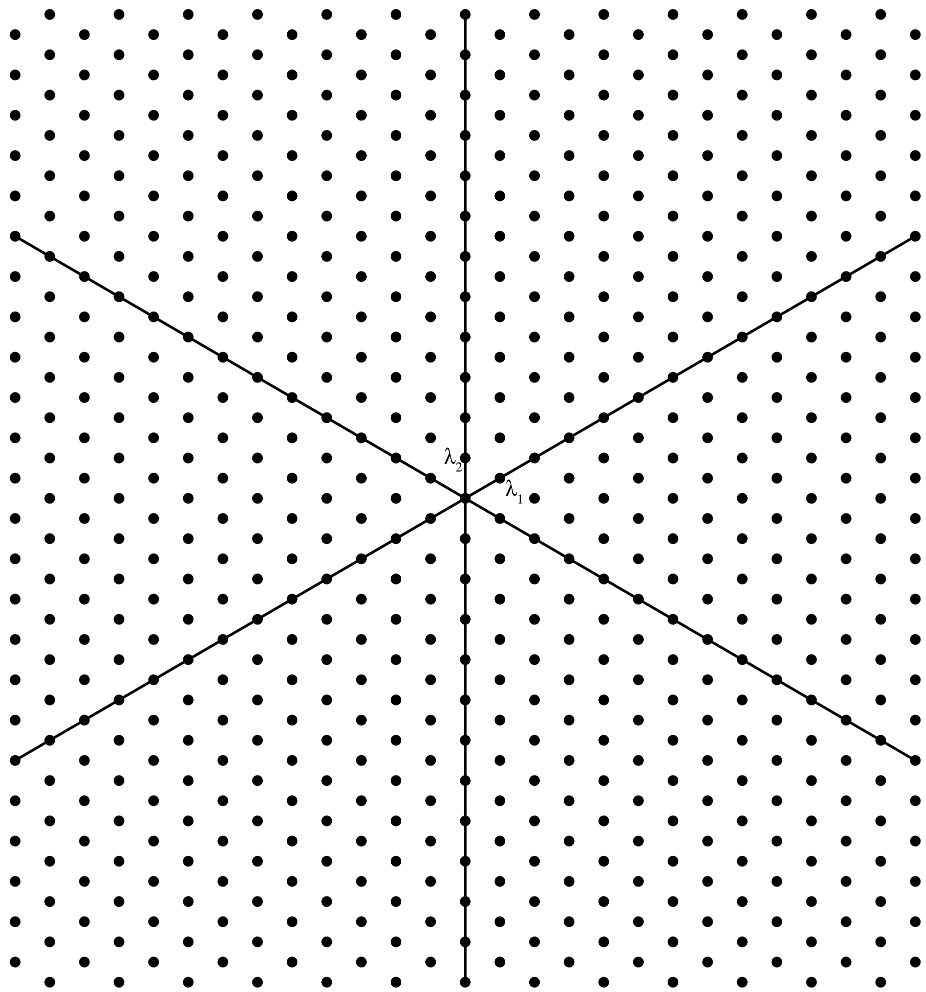


Figure 2: Weight Lattice of Type B_2

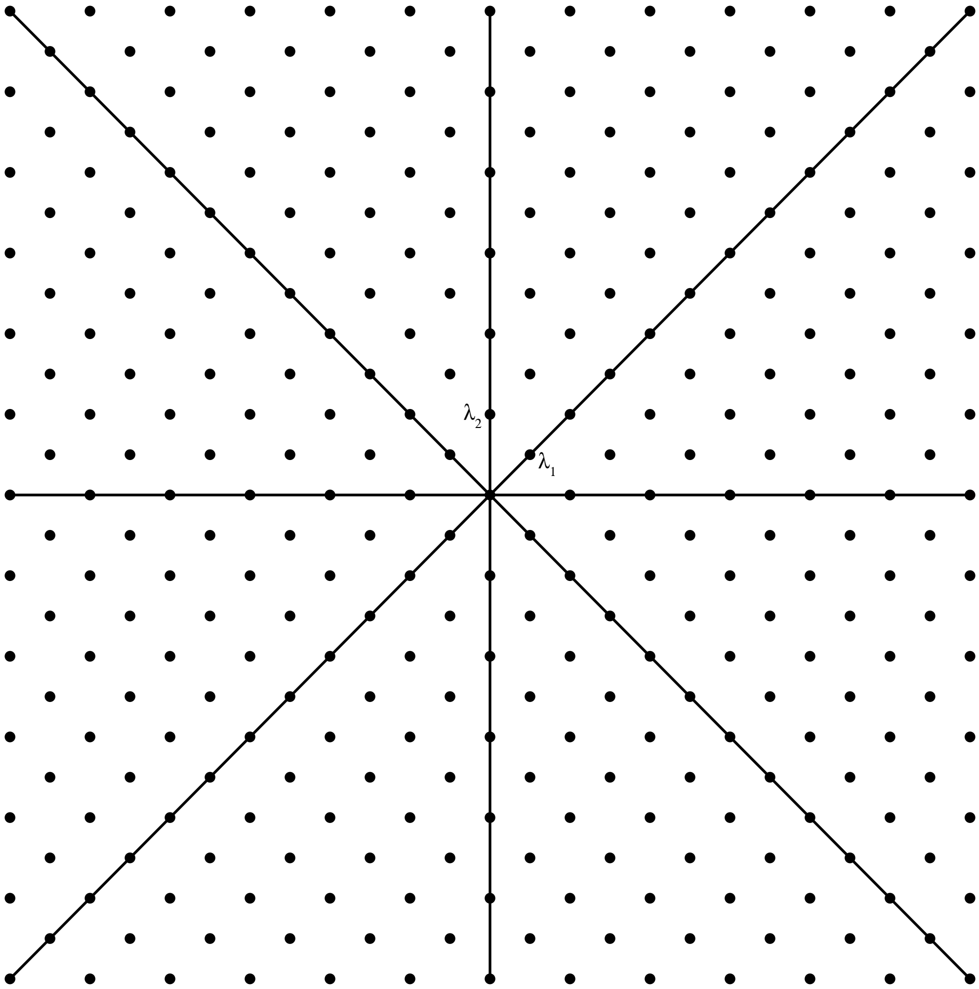


Figure 3: Weight Lattice of Type G_2

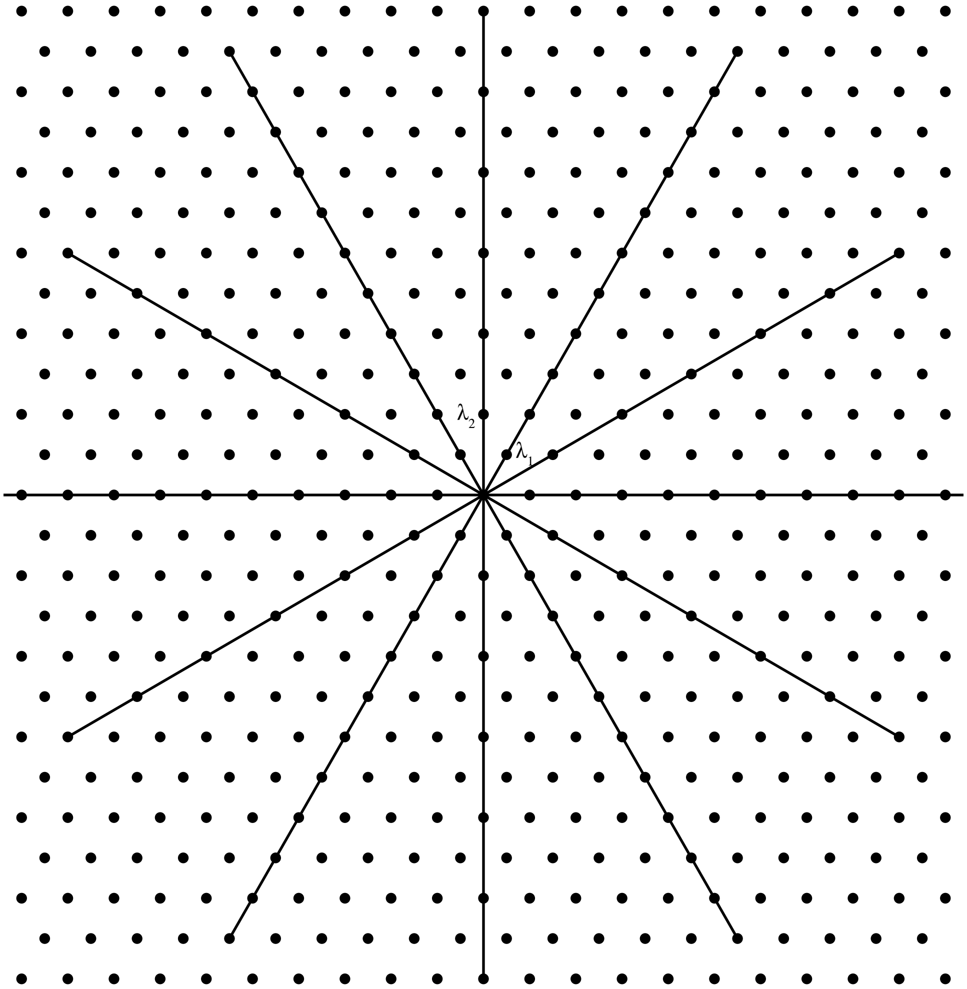


Figure 4: Reflection Lines of Type A_2

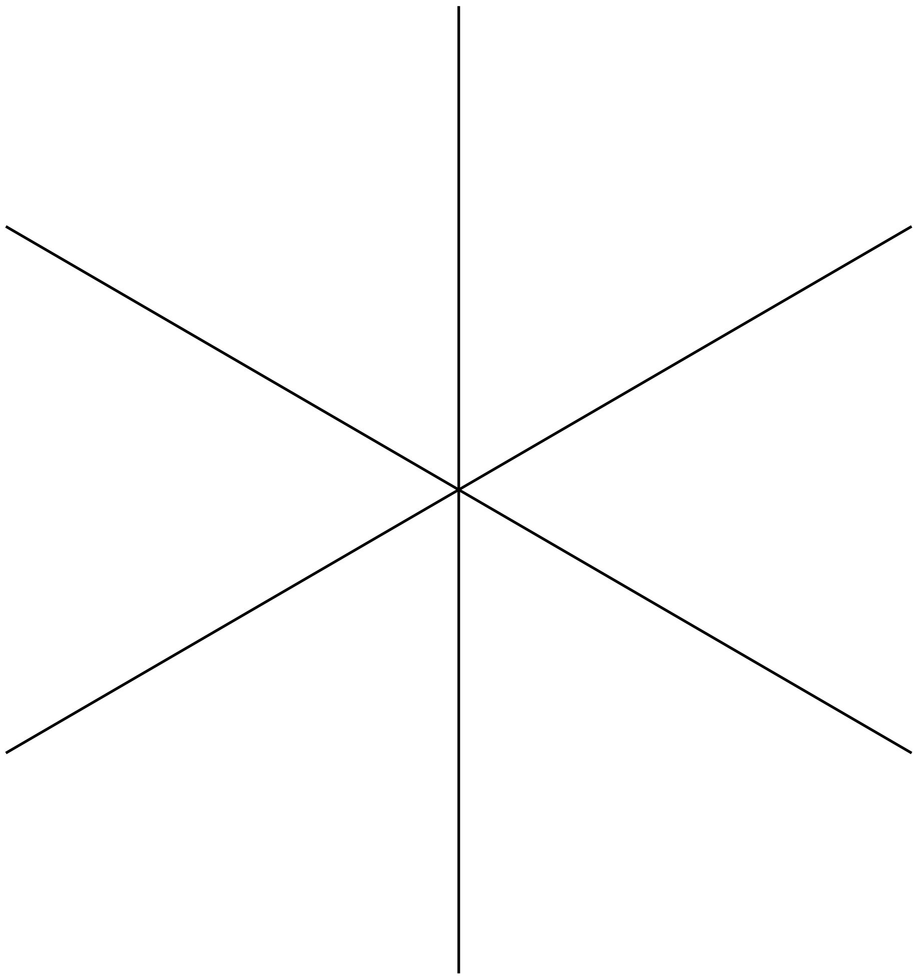


Figure 5: Reflection Lines of Type B_2

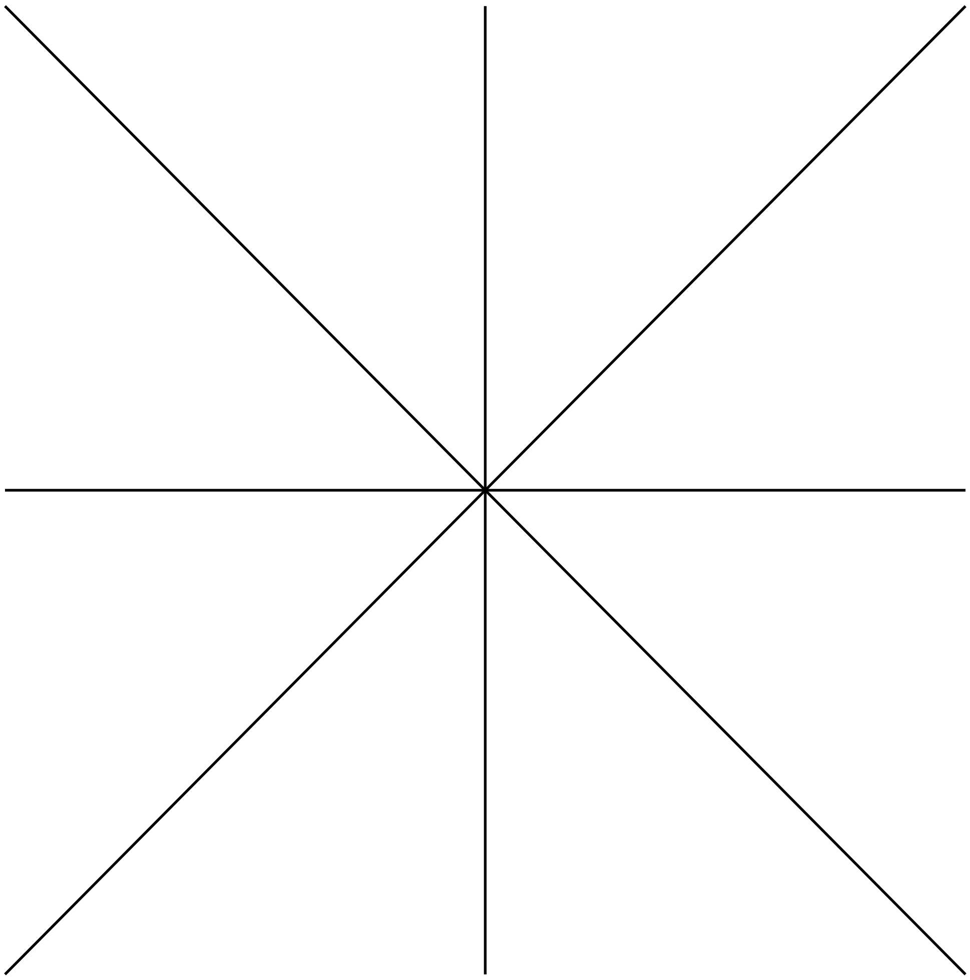


Figure 6: Reflection Lines of Type G_2

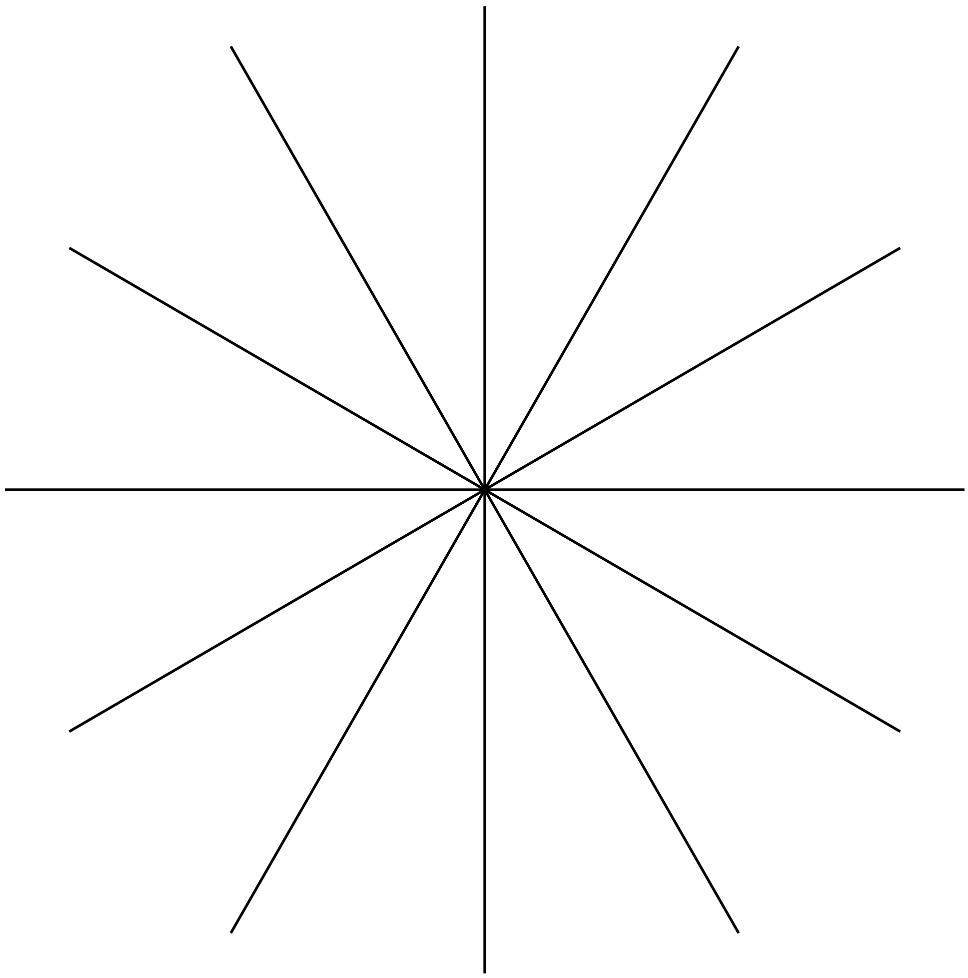


Figure 7: Weyl Conjugates of ρ To Use In Racah Recursion

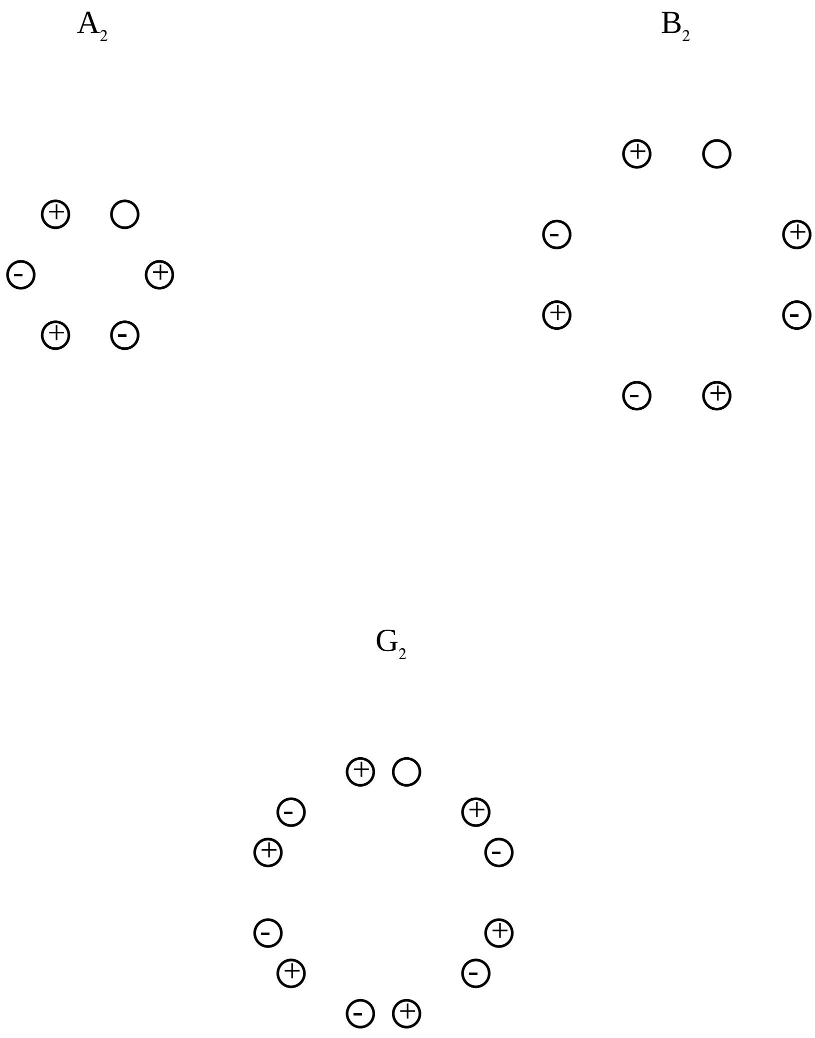


Figure 8: A_2 Weight Diagram For Irreducible Module
With Highest Weight $3\lambda_1 + 2\lambda_2$

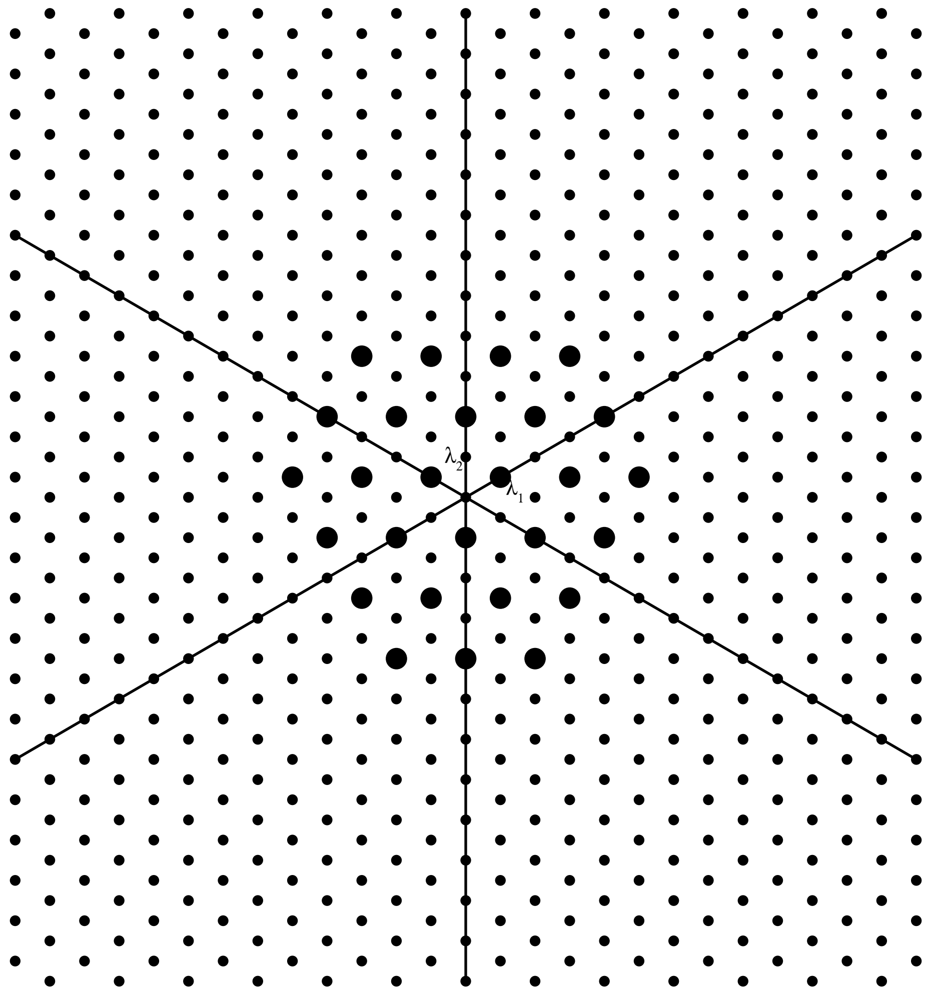


Figure 9: A_2 Weight Diagram For Irreducible Module
With Highest Weight $3\lambda_1 + 2\lambda_2$ Shifted by $\mu + \rho = 2\lambda_1 + 1\lambda_2$

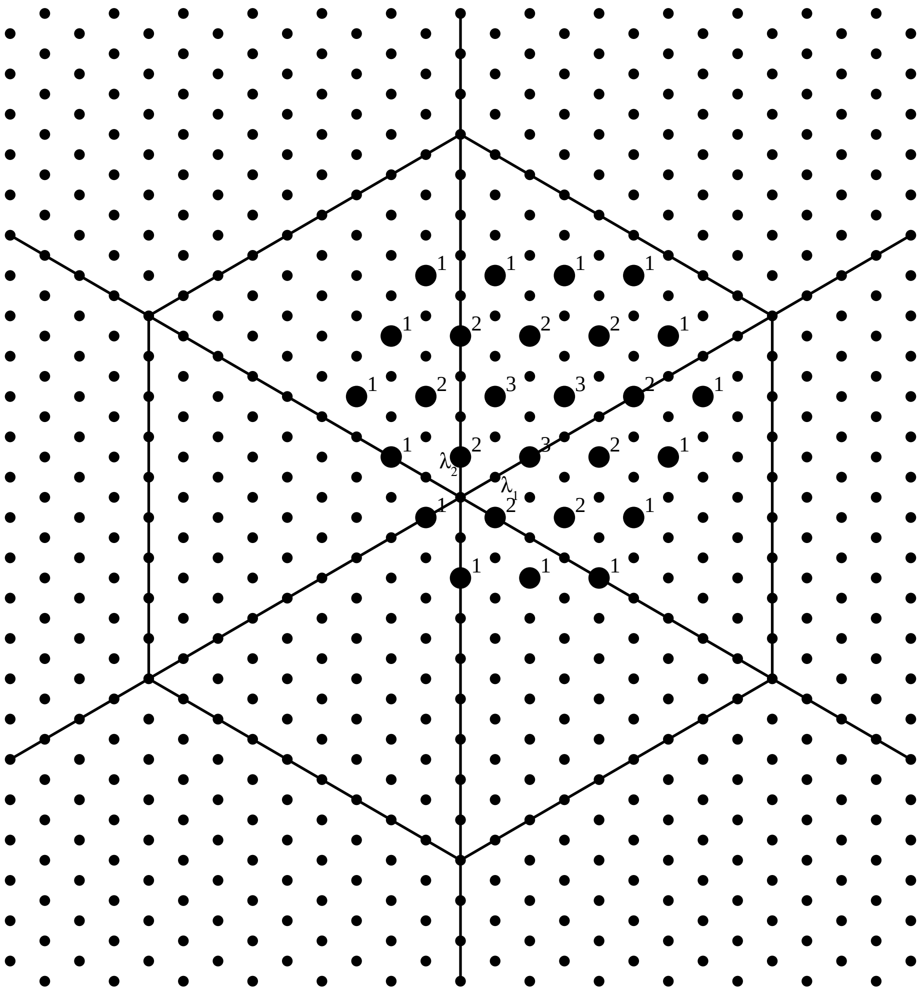


Figure 10: A_2 Weight Diagram Shifted For The
Level 2 Fusion Rule Computation $[3] \cdot [3] = [0] + [3]$

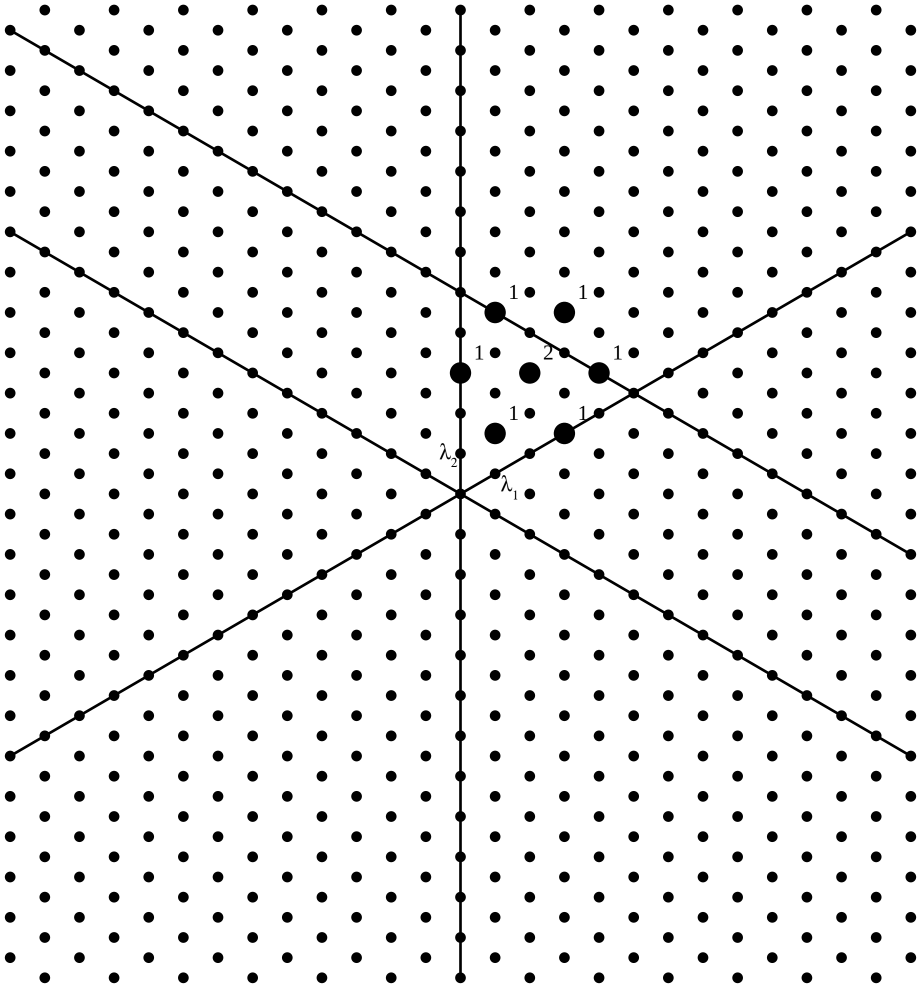


Figure 11: B_2 Weight Diagram Shifted For The Level 1 Fusion Rule Computation $[1] \cdot [1] = [0]$

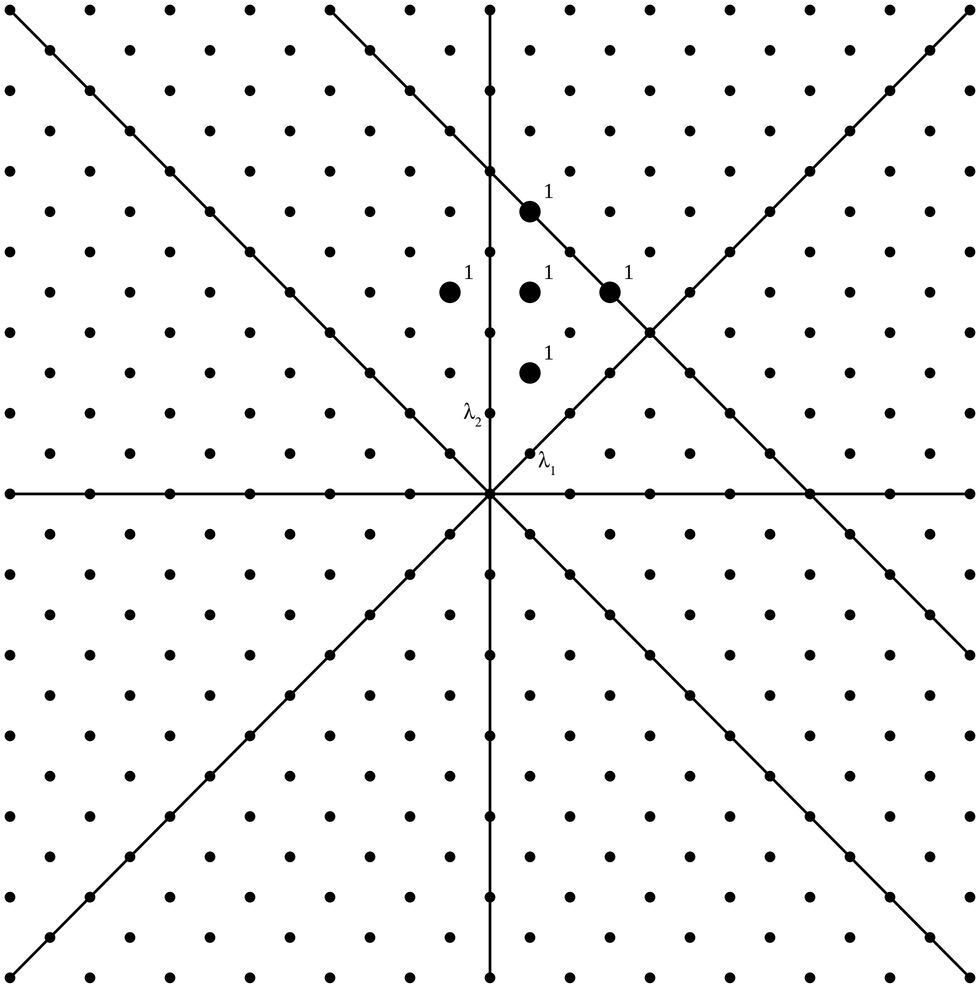


Figure 12: B_2 Weight Diagram Shifted For The
Level 1 Fusion Rule Computation $[1] \cdot [2] = [2]$

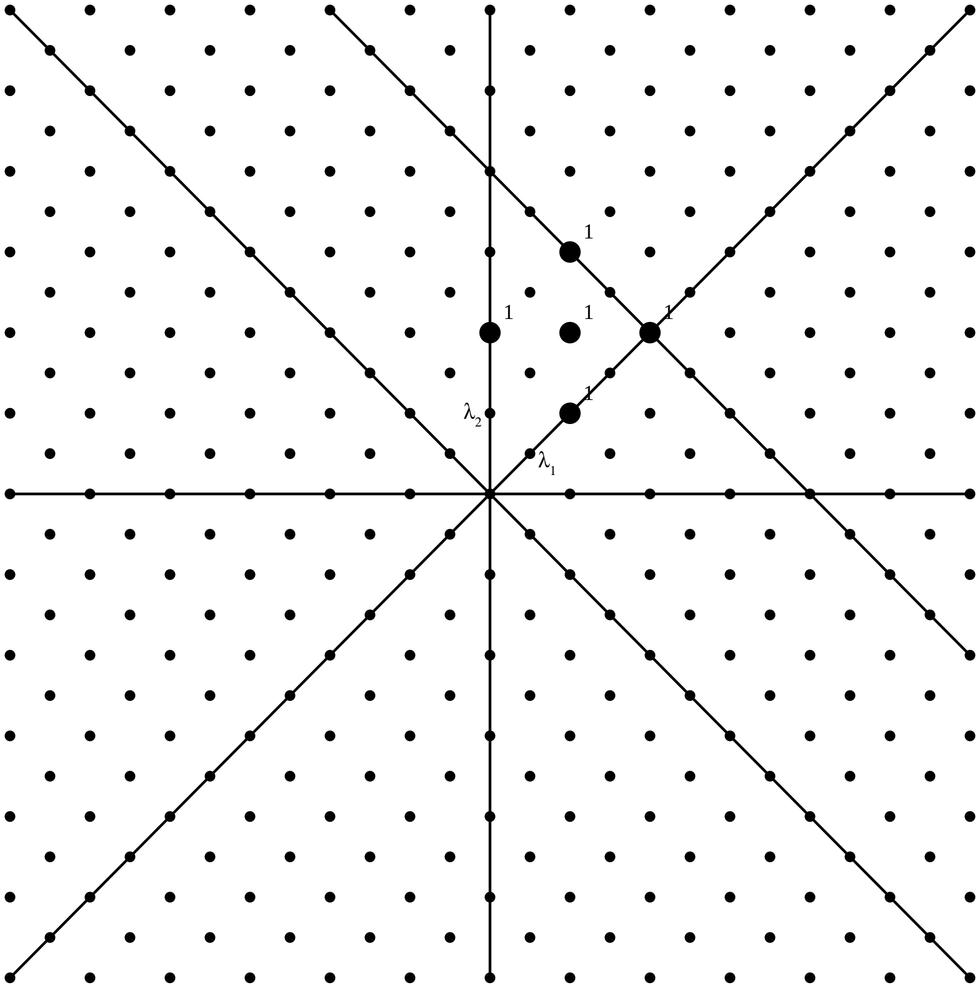


Figure 13: B_2 Weight Diagram Shifted For The Level 1 Fusion Rule Computation $[2] \cdot [2] = [0] + [1]$

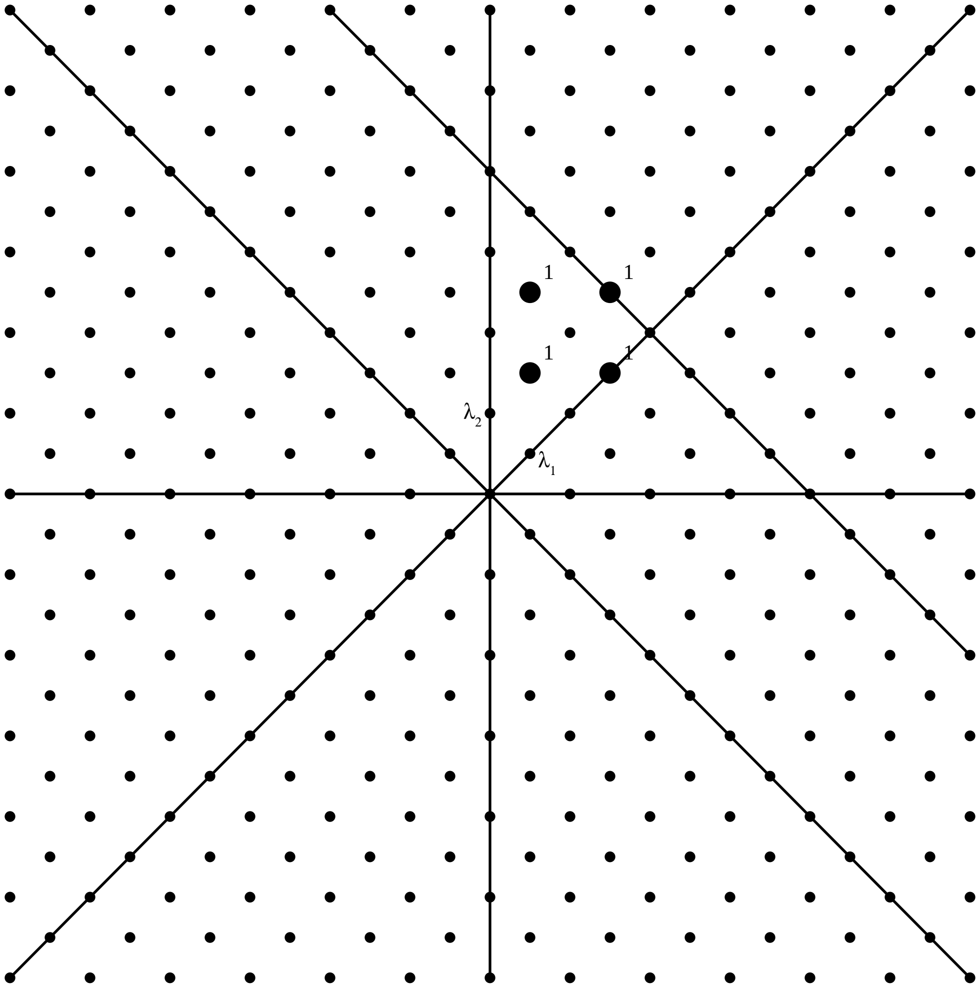


Figure 14: B_2 Weight Diagram Shifted For The
Level 2 Fusion Rule Computation $[3] \cdot [3] = [0] + [4] + [5]$

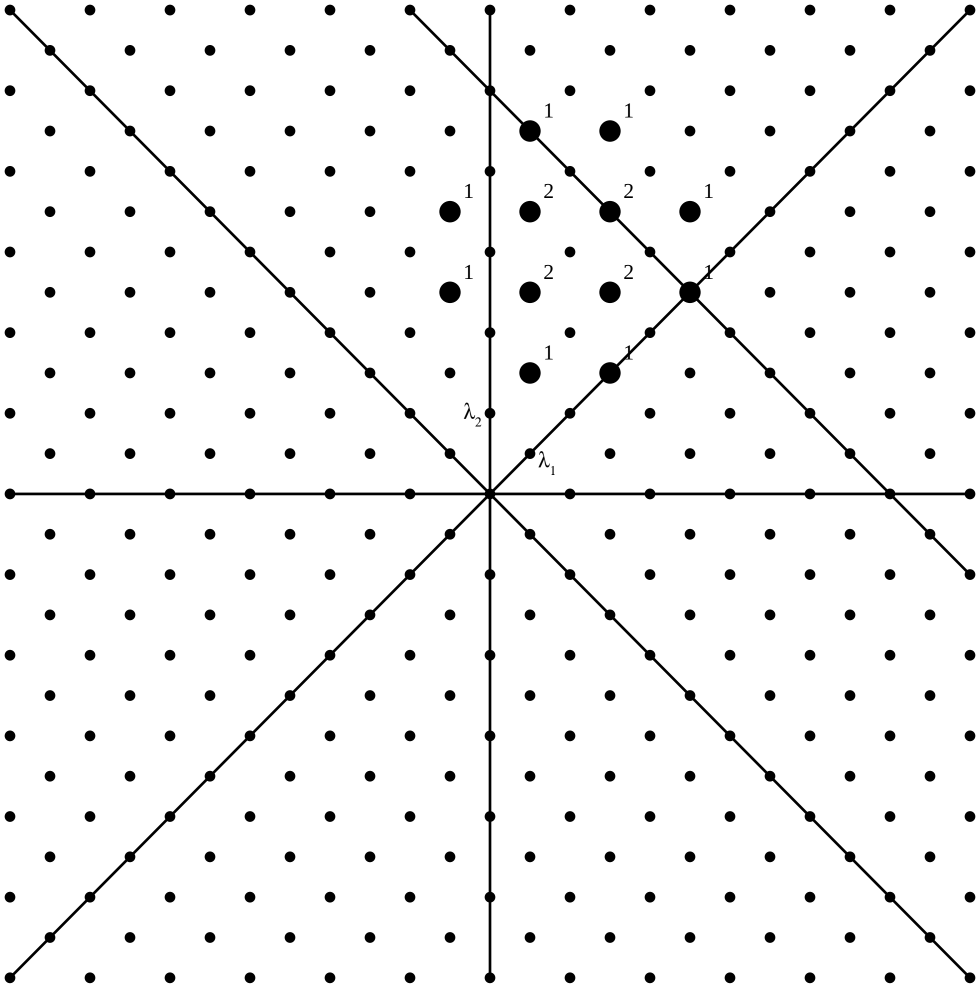


Figure 15: A_2 Weight Diagram For Irreducible Module With Highest Weight $3\lambda_1 + 2\lambda_2$ Shifted by $\lambda_1 + \rho$ For Level 5 Fusion Rule Computations

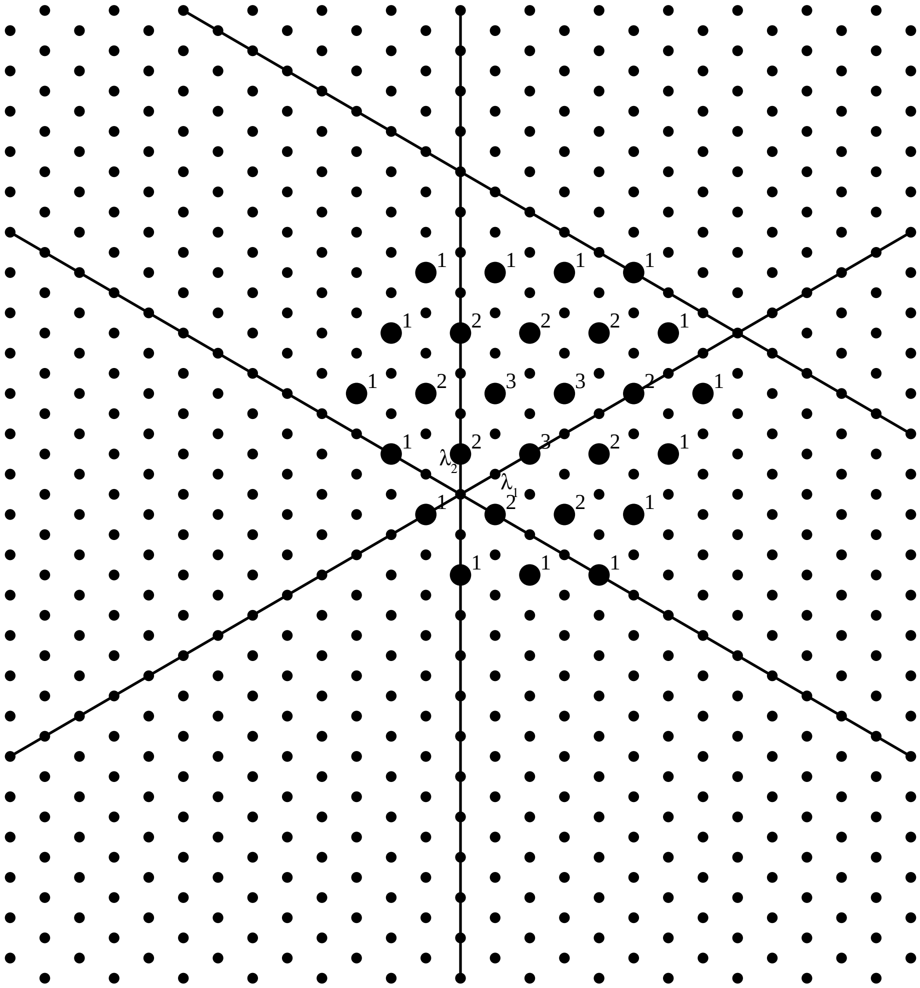


Figure 16: A_2 Weight Diagram For Irreducible Module With Highest Weight λ_1 Shifted by $3\lambda_1 + 2\lambda_2$ For Level 5 Fusion Rule Computations

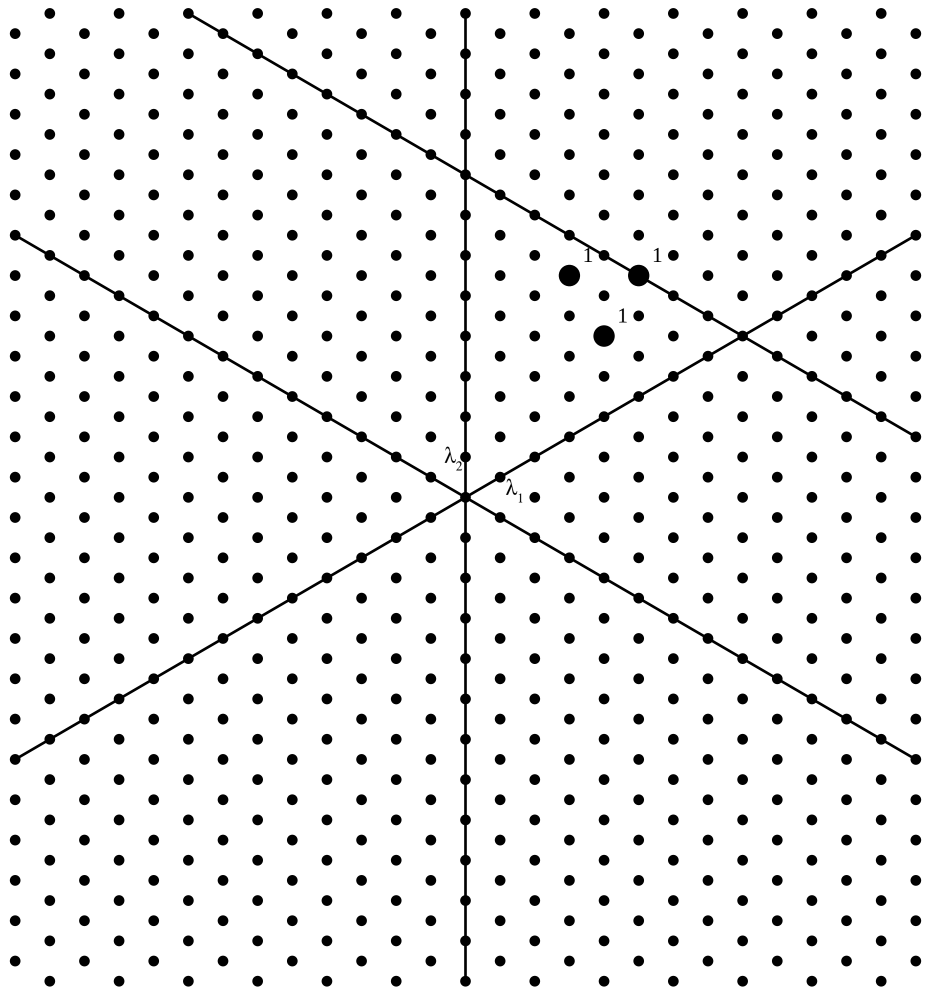


Figure 17: A_2 Weight Diagram For Irreducible Module With Highest Weight $3\lambda_1 + 2\lambda_2$ Shifted by $5\lambda_1 + \rho$ For Level 5 Fusion Rule Computations

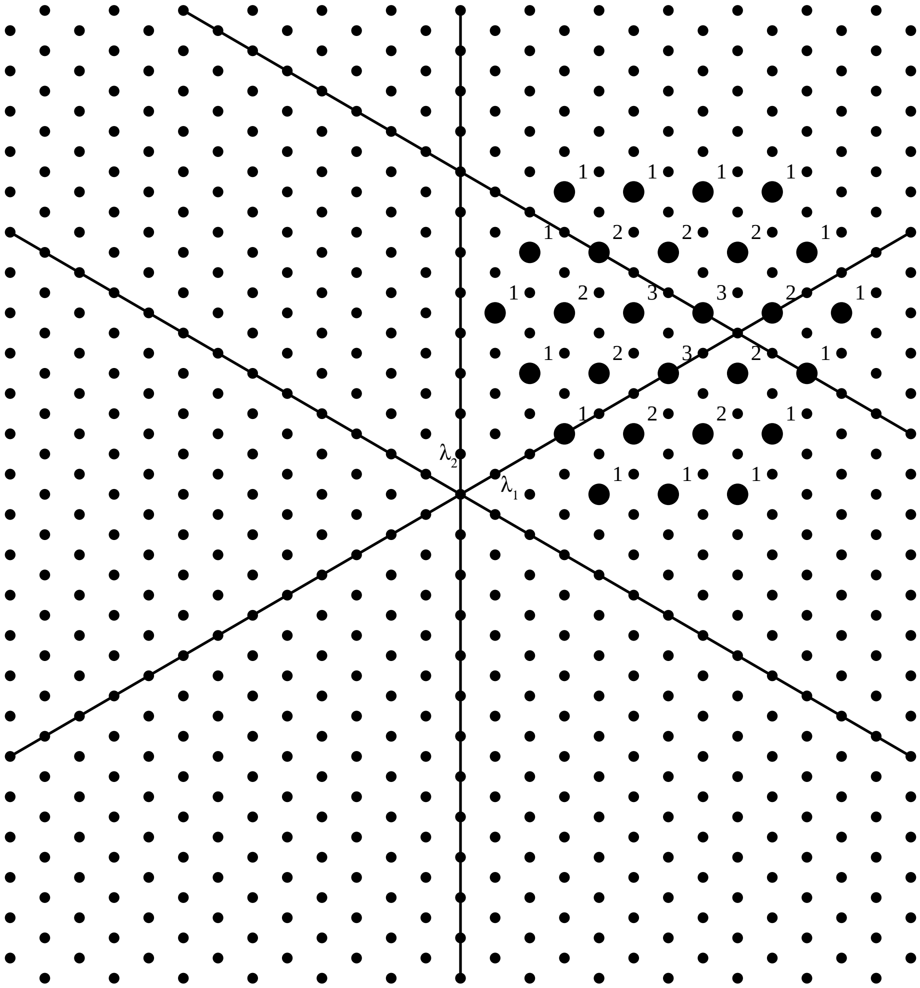
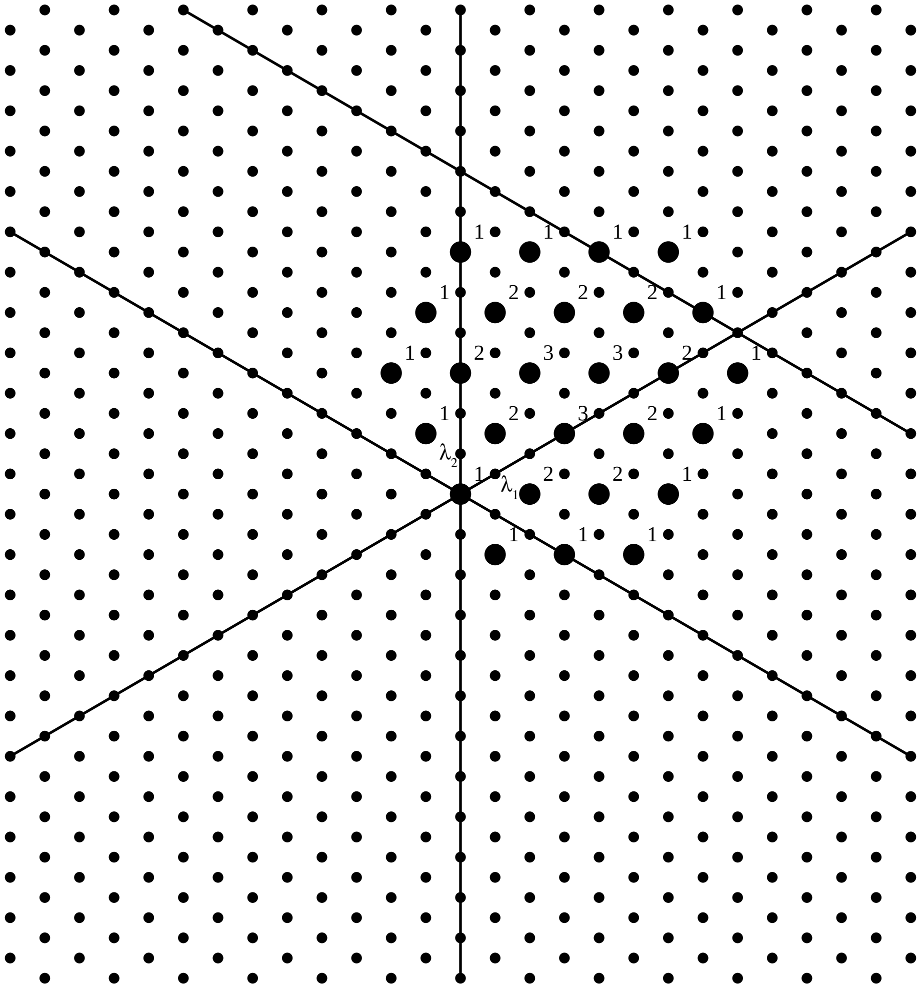


Figure 18: A_2 Weight Diagram For Irreducible Module With Highest Weight $3\lambda_1 + 2\lambda_2$ Shifted by $2\lambda_1 + \rho$ For Level 5 Fusion Rule Computations



References

[AFW] F. Akman, A. Feingold, M. Weiner, *Minimal model fusion rules from 2-groups,* Lett. Math. Phys. **40** (1997), 159–169.

[Bac] P. Bachmann, *Niedere Zahlentheorie*, Chelsea Publ. Co., Bronx, NY 1968, originally published in Leipzig, 1902 (Vol. 1) and 1910 (Vol. 2).

[BMW] L. Bégin, P. Mathieu, M. A. Walton, $\widehat{su}(3)_k$ *fusion coefficients,* Modern Physics Letters **A7** (1992), 3255–3266.

[BKMW] L. Bégin, A. N. Kirillov, P. Mathieu, M. A. Walton, *Berenstein-Zelevinsky triangles, elementary couplings and fusion rules,* Lett. Math. Phys. **28** (1993), 257–268.

[EJP] A. Elashvili, M. Jibladze, D. Pataraia, *Combinatorics of necklaces and "Hermite Reciprocity"*, J. Alg. Comb. **10** (1999), 173–188.

[F1] A. J. Feingold *Zones of uniform decomposition in tensor products,* Proc. Amer. Math. Soc. **70** (1978), 109–113.

[F2] A. J. Feingold *Tensor products of finite dimensional modules for complex semisimple Lie algebras,* Lie Theories and Their Applications, Proceedings of the 1977 Annual Seminar of the Canadian Mathematical Congress, Queen's Papers in Pure and Applied Mathematics, No. 48, (A. J. Coleman and P. Ribenboim, eds.), Queen's University, Kingston, Ontario, 1978, pp. 394–397.

[FFR] A. J. Feingold, I. B. Frenkel, J. F. X. Ries, *Spinor Construction of Vertex Operator Algebras, Triality and* $E_8^{(1)}$, Contemp. Math. **121**, Amer. Math. Soc., Providence, RI, 1991.

[FRW] A. Feingold, J. F. X. Ries, M. D. Weiner, *Spinor construction of the* $c = \frac{1}{2}$ *minimal model*, Moonshine, The Monster, and Related Topics, Contemporary Math. **193**, (Chongying Dong and Geoffrey Mason, eds.), Amer. Math. Soc., Providence, RI, 1995, pp. 45–92.

[FW] A. Feingold, M. Weiner, *Type A fusion rules from elementary group theory,* Proceedings of the Conference on Infinite-Dimensional Lie Theory and Conformal Field Theory, S. Berman, P. Fendley, Y. Huang, K. Misra, and B. Parshall, Editors, Contemporary Mathematics, Vol. 297, Amer. Math. Soc, Providence, RI, 2002.

[FHL] I. B. Frenkel, Yi-Zhi Huang, J. Lepowsky, *On axiomatic approaches to vertex operator algebras and modules*, Memoirs Amer. Math. Soc., 104, No. 594, Amer. Math. Soc., Providence, RI, 1993.

[FLM] I. B. Frenkel, J. Lepowsky, A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Applied Math., 134, Academic Press, Boston, 1988.

[FZ] I. B. Frenkel, Y. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J. **66** (1992), 123–168.

[Fu] J. Fuchs, *Fusion rules in conformal field theory*, Fortsch. Phys. **42** (1994), 1–48.

[GW] D. Gepner, E. Witten *String theory on group manifolds*, Nuclear Physics **B278** (1986), 493–549.

[GO] P. Goddard, D. I. Olive, *Kac-Moody and Virasoro algebras in relation to quantum physics*, Internat. J. Mod. Phys. A, Vol. 1, No. 2 (1986), 303–414.

[Kac] V. G. Kac, *Infinite Dimensional Lie Algebras*, Cambridge University Press, Third Edition, Cambridge, 1990.

[KR] V. G. Kac, A. K. Raina, *Bombay Lectures on Highest Weight Representations*, World Scientific, Singapore, 1987.

[PRV] K. R. Parthasarathy, R. Ranga Rao, V. S. Varadarajan, *Representations of complex semi-simple Lie groups and Lie algebras*, Annals of Mathematics, 2nd Series, Vol. 85 (1967), 383–429.

[Ram] K. G. Ramanathan, *Some applications of Ramanujan's trigonometrical sum* $C_m(n)$, Indian Acad. Sci. Sect. A **20** (1944), 62–70.

[T] G. Tudose, *A special case of* $sl(n)$*-fusion coefficients*, ArXiv:math.CO/0008034 (2000).

[vonS] R. D. von Sterneck, *Ein Analogon zur additiven Zahlentheorie*, Sitzber. Akad. Wiss. Math. Naturw. Class. **111** (1902), 1567–1601.

[Wal] M. A. Walton, *Algorithm for WZW fusion rules: a proof*, Phys. Lett. **B241** (1990), No. 3, 365–368.

[Wal2] M. A. Walton, *Tensor products and fusion rules*, Canadian Journal of Physics **72** (1994), 527–536.

[Wa] Weiqiang Wang, *Rationality of Virasoro vertex operator algebras*, Duke Math. J. IMRN, Vol. 71, No. 1 (1993), 197–211.

Dept. of Math. Sci., The State University of New York, Binghamton, New York 13902-6000

E-mail address: alex@math.binghamton.edu