Subalgebras of Hyperbolic Kac-Moody Algebras

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We dedicate this work to the memory of our friend Peter Slodowy.

Abstract. The hyperbolic (and more generally, Lorentzian) Kac-Moody (KM) Lie algebras $\mathcal{A}$ of rank $r+2>2$ are shown to have a rich structure of indefinite KM subalgebras which can be described by specifying a subset of positive real roots of $\mathcal{A}$ such that the difference of any two is not a root of $\mathcal{A}$. Taking these as the simple roots of the subalgebra gives a Cartan matrix, generators and relations for the subalgebra. Applying this to the canonical example of a rank 3 hyperbolic KM algebra, $\mathcal{F}$, we find that $\mathcal{F}$ contains all of the simply laced rank 2 hyperbolics, as well as an infinite series of indefinite KM subalgebras of rank 3. It is shown that $\mathcal{A}$ also contains Borcherds algebras, obtained by taking all of the root spaces of $\mathcal{A}$ whose roots are in a hyperplane (or any proper subspace). This applies as well to the case of rank 2 hyperbolics, where the Borcherds algebras have all their roots on a line, giving the simplest possible examples.

1. Introduction

More than thirty years after their discovery, indefinite and, more specifically, hyperbolic Kac-Moody (KM) algebras remain an unsolved challenge of modern mathematics. For instance, there is not a single such algebra for which the root multiplicities are known in closed form. Even less is known about the detailed structure of the imaginary root spaces, and a manageable explicit realization analogous to the current algebra realization of affine algebras appears to be beyond reach presently. (However, there are intriguing hints that a "physical" realization might be found in an extension of Einstein's theory of general relativity, see [**DHN**] and references therein). Given this lack of knowledge, any new information about these algebras may be of potential value and importance.

In this note we point out that hyperbolic KM algebras possess a very rich structure of nonisomorphic infinite dimensional subalgebras, all of which can be generated by a set of root vectors E_α and F_α, where certain roots α from the root system of the original algebra are taken as new simple roots. If these roots

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are real, then the subalgebra can be an indefinite KM algebra; however, we will also exhibit examples where the new simple roots are imaginary, and the resulting subalgebra is a Borcherds algebra [**B1, Jur1, Nie**]. Two subalgebras are said to be W-equivalent if their respective simple roots are related by a Weyl transformation from the Weyl group of the original algebra. Geometrically, some of the non-W-equivalent subalgebras of indefinite KM algebras can be understood in terms of conical sections. Generally there are three ways of slicing indefinite (and hyperbolic) KM algebras corresponding to the intersections of the light-cone with a hyperplane in the dual of the Cartan subalgebra. That dual space is equipped with an indefinite bilinear form, (α, β), and depending on whether the hyperplane is space-like, light-like or time-like (meaning that its normal vector, ν, satisfies $\nu^2 = (\nu, \nu) < 0$, $\nu^2 = 0$, or $\nu^2 > 0$) one obtains elliptic, parabolic or hyperbolic sections corresponding to a decomposition of the original KM algebra with respect to a finite, an affine or an indefinite subalgebra. (The latter need not be hyperbolic, even if the original algebra is hyperbolic.)

Previous attempts to understand indefinite KM algebras were almost exclusively based on a decomposition into irreducible representations of their affine subalgebras. (Decompositions with respect to finite subalgebras seem to have received scant attention so far.) Although the theory of representations of affine algebras is reasonably well understood, this approach has not yielded much information about the behavior of the algebra "deep within the light-cone". A complete analysis was carried out up to affine level two in [**FF**] for the rank 3 hyperbolic algebra, $\mathcal{F}$ (also referred to as $HA_1^{(1)}$), containing the affine algebra $A_1^{(1)}$. This work was generalized in [**KMW**] to the rank 10 hyperbolic KM algebra E_{10}, where the first two levels with respect to the affine subalgebra $E_9 = E_8^{(1)}$ were analyzed. This analysis has been extended as far as level 5 for $\mathcal{F}$ and also carried out for some levels for some other algebras [**BKM1, BKM2, BKM3, Ka1, Ka2, Ka3, Ka4, Ka5, KaM1, KaM2**], but appears rather hopeless as a method for giving all levels. Finally, DDF operators have been used in [**GN**] to probe some root spaces in detail.

We conclude this introduction with a brief description of the main results in each section. In section 2 we give the necessary background and details about the rank 3 hyperbolic KM algebra, $\mathcal{F}$, which the most decisive example for understanding the subject. In section 3 we prove the key theorem which identifies a class of subalgebras of KM algebras by the choice of a set of positive real roots whose differences are not roots. We apply that theorem to the algebra $\mathcal{F}$ to find inside it all simply laced rank 2 hyperbolic KM algebras, as well as an infinite series of inequivalent rank 3 indefinite KM algebras. We have generalizations of these results where the role of the algebra $\mathcal{F}$ is played by an arbitrary rank $r+2$ Lorentzian KM algebra, $\mathcal{A}$, and we find two infinite series of inequivalent indefinite KM subalgebras, one series of rank $r+1$ and one series of rank $r+2$. In section 4 we go back to the special case of $\mathcal{F}$, and exploit the beautiful geometry of its Weyl group, W, which is the hyperbolic triangle group $T(2,3,\infty) = PGL_2(\mathbf{Z})$, through its action on the Poincaré disk model of the hyperbolic plane. This allows us to find, in principle, a large class of KM subalgebras of $\mathcal{F}$ which are inequivalent to those already found in section 3. A complete classification of such subalgebras is beyond the scope of this paper, but we give a number of representative examples to illustrate the idea. We also discuss how the geometry of the Poincaré disk gives information about the generators and relations defining the Weyl groups of these subalgebras, and how they are related to the Weyl group W. We then use this geometrical picture to relate the Weyl groups of the series of rank 3 indefinite subalgebras of $\mathcal{F}$ found in section 3 to W. (Thanks to Tadeusz Januszkiewicz for suggesting and explaining this geometrical way of finding subgroups of W.) In section 5 we give a method of finding Borcherds generalized KM algebras inside $\mathcal{F}$, inside and Lorentzian algebra $\mathcal{A}$, and even inside rank 2 hyperbolics.

2. The rank 3 algebra $\mathcal{F}$

The canonical example of a hyperbolic KM algebra is the rank 3 algebra $\mathcal{F}$ studied in [**FF**], whose Cartan matrix is

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} ,$$

whose simple roots are α_{-1}, α_0, α_1, and whose Dynkin diagram is

$$\underset{\alpha_{-1}}{\bullet} \!-\!\!-\!\!-\! \underset{\alpha_0}{\bullet} \equiv\!\equiv \underset{\alpha_1}{\bullet}\ .$$

This algebra is the minimal rank hyperbolic KM algebra with both finite and affine KM subalgebras. Rank 2 hyperbolic KM algebras are also quite interesting ([**F, KaM2**]) because of their connections with real quadratic fields and Hilbert modular forms [**LM**]. They have infinitely many A_1 subalgebras, infinitely many rank 2 hyperbolic KM subalgebras, and our new work shows that they contain Borcherds subalgebras as well. But our main inspiration has been the hyperbolic algebra $\mathcal{F}$, which seems to be the simplest such algebra that incorporates all the essential features of higher rank examples. Further evidence that $\mathcal{F}$ deserves most serious study is that it contains all the symmetric rank 2 hyperbolics, as well as many other rank 3 hyperbolics. From a physicist's point of view, it is attractive because it may appear as a hidden symmetry in (an extension of) Einstein's theory of gravity in four space-time dimensions (see [**DHN**] for a review of recent developments and further references).

Many interesting results about the structure of $\mathcal{F}$ were obtained in [**FF**], as well as a relationship with Siegel modular forms of genus 2, but a complete understanding of the root multiplicities has remained elusive despite considerable further work ([**BKM1, Ka3, Ka4, Ka5, KaM1**]). It is apparent from the Cartan matrix and Dynkin diagram that $\mathcal{F}$ contains an affine subalgebra $\mathcal{F}_0$ of type $A_1^{(1)}$ with simple roots α_0, α_1. The approach in [**FF**] is based on the decomposition

$$\mathcal{F} = \bigoplus_{n\in\mathbf{Z}} \mathcal{F}_n$$

with respect to $\mathcal{F}_0$, that is, with respect to the "level" which is the value of the central element of $\mathcal{F}_0$. The feature of $\mathcal{F}$ which first attracted the attention of AJF was its Weyl group, W, which is a reflection group

$$\langle r_{-1}, r_0, r_1 \mid r_{-1}^2 = r_0^2 = r_1^2 = 1, (r_{-1}r_0)^3 = 1, (r_{-1}r_1)^2 = 1\rangle$$

isomorphic to the hyperbolic triangle group, $T(2,3,\infty)$ and to $PGL_2(\mathbf{Z})$. The action of W shows that there are infinitely many subalgebras of type $A_1^{(1)}$ inside $\mathcal{F}$, corresponding to cusps of the modular group, or to lines of null roots on the light-cone. However, these lines of null roots are all related by the Weyl group, and therefore the corresponding affine subalgebras of $\mathcal{F}$ are all W-equivalent.

To give more details, let H be the Cartan subalgebra of $\mathcal{F}$ and let H^* be its dual space, which has a basis consisting of the simple roots $\{\alpha_{-1}, \alpha_0, \alpha_1\}$, and a symmetric bilinear form whose matrix with respect to this basis is the Cartan matrix above. That form is indefinite, and can be very conveniently described as follows. On the space $S_2(\mathbf{R})$ of 2×2 symmetric real matrices define a bilinear form by

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ b' & c' \end{pmatrix} = 2bb' - ac' - a'c$$

so the associated quadratic form is

$$2b^2 - 2ac = -2 \det \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Then the root lattice of $\mathcal{F}$ is isometric to $S_2(\mathbf{Z})$ where the entries a, b and c are integers, and the weight lattice of $\mathcal{F}$ is isometric to $S_2'(\mathbf{Z})$ where a and c are integers but $b \in \frac{1}{2}\mathbf{Z}$. We make the correspondence explicit by choosing

$$\alpha_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \alpha_0 = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Weyl group action of $A \in PGL_2(\mathbf{Z})$ on $N = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is given by $A(N) = ANA^t$ and the explicit matrices which represent the three simple reflections are

$$r_{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad r_0 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad r_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The root system of $\mathcal{F}$ is

$$\Phi = \{N \in S_2(\mathbf{Z}) \mid \det(\mathbf{N}) \geq -\mathbf{1}\}$$

which is just the elements N of the root lattice for which the norm squared is less than or equal to 2, $-2\det(N) \leq 2$. The real roots (Weyl conjugates of the simple roots) are

$$\Phi_{real} = \{N \in S_2(\mathbf{Z}) \mid \det(\mathbf{N}) = -\mathbf{1}\}$$

which lie on a single sheeted hyperboloid. The light-cone is the set of points $N \in S_2(\mathbf{R})$ where $\det(N) = 0$. All real roots have multiplicity one, and in the case of $\mathcal{F}$, the same is true of all roots on the light-cone. This comes from $\mathcal{F}_0$ (with simple roots α_0 and α_1) whose underlying finite dimensional Lie algebra is the rank one Lie algebra sl_2 of type A_1. Any light-cone root, $N \in \Phi$, satisfies $\det(N) = 0$ and is W equivalent to a unique one on level 0 of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = -a(\alpha_0 + \alpha_1)$ with $0 \neq a \in \mathbf{Z}$. The roots Φ decompose into the disjoint union of positive and negative roots, and we can distinguish these level 0 light-cone roots easily according to whether a is negative or positive. The roots on level 1 are of the form $\begin{pmatrix} a & b \\ b & 1 \end{pmatrix} = -(a+1)(\alpha_0 + \alpha_1) + b\alpha_1 - \alpha_{-1}$ with $a - b^2 \geq -1$ so $a \geq -1$. The affine Weyl group of $\mathcal{F}_0$ is an infinite dihedral group generated by r_0 and r_1, and it preserves each level. Each root on level 1 is equivalent by the affine Weyl group to one of the form $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = -(a+1)(\alpha_0 + \alpha_1) - \alpha_{-1}$, and because level 1, $\mathcal{F}_1$, is a single irreducible $\mathcal{F}_0$-module (the basic module), the multiplicity of that root is a value of the classical partition function, $p(a+1)$ [**FL**]. Each root on level 2 is equivalent by the affine Weyl group to one of the form $\begin{pmatrix} a & 0 \\ 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} a & 1 \\ 1 & 2 \end{pmatrix}$. The $\mathcal{F}_0$-module structure of level 2 is much more subtle, being the antisymmetric tensors in $\mathcal{F}_1 \otimes \mathcal{F}_1$ with one irreducible module removed. If we form a generating function of the level 2 root multiplicities in those two columns,

$$\sum_{a=0}^{\infty} \mathrm{Mult} \begin{pmatrix} a & 1 \\ 1 & 2 \end{pmatrix} \ q^{2a} + \mathrm{Mult} \begin{pmatrix} a & 0 \\ 0 & 2 \end{pmatrix} \ q^{2a+1}$$

it is remarkable that the first 20 coefficients are again given by the partition function $p(\det(N)+1)$. The exact formula for that generating function

$$\left[\sum_{n\geq 0} p(n) q^n\right] \left[\prod_{j\geq 1} (1-q^{4j-2})\right] \frac{q^{-3}}{2} \left[\prod_{j\geq 1}(1+q^{2j-1}) - \prod_{j\geq 1}(1-q^{2j-1}) - 2q\right]$$

was one of the main results of [**FF**], but further work has shown the increasing complexity of higher levels as $\mathcal{F}_0$-modules.

There are other ways in which one might naturally decompose $\mathcal{F}$, for example, with respect to the finite dimensional subalgebra of type A_2 with simple roots α_{-1} and α_0. In that decomposition the analogue of level of a root $\alpha = \sum a_i \alpha_i$ would be the coefficient a_1 of α_1. Each graded piece of the decomposition would be a finite dimensional A_2 module, and it is not hard to use a computer to find the irreducible modules which occur and their multiplicities for quite a few levels[1]. The question is whether there is a useful (recursive) description of those module multiplicities which sheds more light on the hyperbolic root multiplicities.

3. Indefinite subalgebras from subroot systems

The main point of the present contribution is that indefinite KM algebras possess infinitely many non-isomorphic subalgebras, which are of indefinite type (in fact, KM subalgebras of equal rank with inequivalent Cartan matrices). Unlike finite or affine subalgebras, these subalgebras are themselves not yet well understood, but the corresponding decompositions may nevertheless provide valuable new viewpoints.

We will use the following theorem, which allows us to find subalgebras by locating a set of simple roots for the subalgebra within the root system of the larger algebra. One cannot just choose an arbitrary subset of positive roots and declare them to be the simple roots of a subalgebra. For example, if root vectors E_i correspond to simple roots β_i and root vectors F_j correspond to their negatives $-\beta_j$, then one of the required Serre relations, $[E_i, F_j] = \delta_{ij} H_i$ could be violated for $i \neq j$ if $\beta_i - \beta_j$ were a root of the larger algebra.

THEOREM 3.1. *Let Φ be the set of roots of a Kac-Moody Lie algebra, $\mathfrak{g}$, with Cartan subalgebra, $\mathfrak{h}$, and let Φ^+_{real} be the positive real roots of $\mathfrak{g}$. Let $\beta_1, \cdots, \beta_n \in \Phi^+_{real}$ be chosen such that for all $1 \leq i \neq j \leq n$, we have $\beta_i - \beta_j \notin \Phi$. For $1 \leq i \leq n$ let $0 \neq E_i \in \mathfrak{g}_{\beta_i}$ and $0 \neq F_i \in \mathfrak{g}_{-\beta_i}$ be root vectors in the one-dimensional root spaces corresponding to the positive real roots β_i, and the negative real roots $-\beta_i$, respectively, and let $H_i = [E_i, F_i] \in \mathfrak{h}$. Then the Lie subalgebra of $\mathfrak{g}$*

[1]The level decomposition with respect to the coefficient a_1 has been used in [**DHN**], and is known up to level $a_1 = 56$ (T. Fischbacher, private communication).

generated by $\{E_i, F_i, H_i \mid 1 \le i \le n\}$ *is a Kac-Moody algebra with Cartan matrix* $C = [C_{ij}] = [2(\beta_i, \beta_j)/(\beta_j, \beta_j)]$. *Denote this subalgebra by* $\mathfrak{g}(\beta_1, \cdots, \beta_n)$.

PROOF. By construction, the elements H_i are in the Cartan subalgebra, $\mathfrak{h}$, so the following relations are clear:

$$[H_i, E_j] = \beta_j(H_i)E_j, \quad [H_i, F_j] = -\beta_j(H_i)F_j, \quad [H_i, H_j] = 0.$$

For $i \neq j$, the bracket $[E_i, F_j]$ would be in the $\beta_i - \beta_j$ root space, but we have chosen the β_i such that this difference is not a root of $\mathfrak{g}$, so that bracket must be zero, giving the relations

$$[E_i, F_j] = \delta_{ij} H_i.$$

To check that $(ad\ E_i)^{1-C_{ji}}\ E_j = 0$ for all $i \neq j$, it would suffice to show that $(1 - C_{ji})\beta_i + \beta_j$ is not in Φ. Since β_j is a real root of $\mathfrak{g}$, and since $\beta_j - \beta_i \notin \Phi$, the β_i root string through β_j is of the form

$$\beta_j, \beta_j + \beta_i, \cdots, \beta_j + q\beta_i$$

where $-q = 2(\beta_j, \beta_i)/(\beta_i, \beta_i) = C_{ji} \le 0$. Therefore, C satisfies the conditions to be a Cartan matrix and

$$\beta_j + (q+1)\beta_i = \beta_j + (1 - C_{ji})\beta_i$$

is not in Φ. The relations $(ad\ F_i)^{1-C_{ji}}\ F_j = 0$ for all $i \neq j$, follow immediately since $-(1 - C_{ji})\beta_i - \beta_j$ is not in Φ. This shows that all of the Serre relations determined by the Cartan matrix C are satisfied by the generators given above. □

Our first application of this theorem is to show that the rank 3 algebra $\mathcal{F}$ contains all the simply laced rank 2 hyperbolic KM algebras as subalgebras. The decomposition of $\mathcal{F}$ with respect to its affine algebra $\mathcal{F}_0$ corresponded to slicing with respect to planes parallel to an edge of the light-cone, and the affine Weyl group acted on those planes by moving roots along parabolas. Decomposition with respect to the finite dimensional algebra A_2 corresponds to slices which intersect the light-cone in circles or ellipses. So it should come as no surprise that there should be subalgebras whose decompositions correspond to slices which intersect the light-cone in hyperbolas. Consequently we will "locate" the rank 2 hyperbolic KM algebras by identifying their simple root systems inside the root system of $\mathcal{F}$. There are two equally good choices for each algebra, distinguished in the theorem below by a choice of plus or minus sign.

THEOREM 3.2. *Fix any integer* $m \ge 1$. *In the root system* Φ *of* $\mathcal{F}$ *we have the positive roots*

$$\beta_0 = \alpha_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\beta_1 = \beta_1^{(m)} = m(\alpha_0 + \alpha_1) \pm \alpha_1 = \begin{pmatrix} -m & \pm 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Then the KM Lie subalgebra $\mathcal{F}(\beta_0, \beta_1)$ *constructed in Theorem 3.1 has Cartan matrix*

$$\begin{bmatrix} 2 & -m \\ -m & 2 \end{bmatrix}$$

which is rank 2 hyperbolic for $m \ge 3$. *To indicate the dependence on* m *we will denote this subalgebra by* $\mathcal{H}(m)$.

Proof. The result follows from Theorem 3.1 because $\beta_0, \beta_1^{(m)} \in \Phi^+_{real}$ and

$$\beta_1^{(m)} - \beta_0 = m(\alpha_0 + \alpha_1) \pm \alpha_1 - \alpha_{-1} = \begin{pmatrix} -m-1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$$

has determinant $-m-2 < -2$ for $m \geq 1$, so is not a root of $\mathcal{F}$. $\square$

While the real root vector E_0 of $\mathcal{H}(m)$ may be taken to be the simple root generator e_{-1} of $\mathcal{F}$, the real root vector E_1 may be written as the $2m$-fold commutator

$$E_1 = [e_1, [e_0, [e_1, \cdots [e_1, [e_0, e_1] \cdots]]]$$

if $\beta_1^{(m)} = m(\alpha_0 + \alpha_1) + \alpha_1$, and as the commmutator

$$E_1 = [e_0, [e_1, [e_0, \cdots [e_0, [e_1, e_0] \cdots]]]$$

if $\beta_1^{(m)} = m(\alpha_0 + \alpha_1) - \alpha_1$. (Because $\beta_1^{(m)}$ is real, all reorderings of these expressions are equivalent.)

The above theorem generalizes to any Lorentzian KM algebra, $\mathcal{A}$, that is, one which is obtained from an affine algebra by the procedure of "over-extension", attaching an additional node by one line only to the affine node in an affine Dynkin diagram. For any root α such that $(\alpha, \alpha) \neq 0$, define $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. Then the Cartan integers which are the entries of the Cartan matrix of the rank $r+2$ algebra $\mathcal{A}$ are given by $(\alpha_i, \alpha_j^\vee)$.

Theorem 3.3. *Let $\{\alpha_i \mid -1 \leq i \leq r\}$ be the simple roots of a Lorentzian KM algebra, $\mathcal{A}$, so $\alpha_1, \ldots, \alpha_r$ are the simple roots of a finite dimensional simple Lie algebra, α_0 is the affine simple root which generates an affine root system when included in the list, and α_{-1} satisfies $(\alpha_{-1}, \alpha_{-1}) = 2, (\alpha_{-1}, \alpha_0) = -1$, $(\alpha_{-1}, \alpha_i) = 0$ for $1 \leq i \leq r$. Write the affine null root $\delta = \sum_{j=0}^{r} n_j \alpha_j$ where $n_0 = 1$. Fix any integer $m \geq 0$ and define*

$$\beta_0 = \alpha_{-1}, \quad \beta_1 = \beta_1^{(m)} = m\delta + \alpha_1, \quad \beta_j = \alpha_j \quad \text{for } 2 \leq j \leq r.$$

Then the KM Lie subalgebra $\mathcal{A}(\beta_0, \cdots, \beta_r)$ constructed in Theorem 3.1 has Cartan matrix

$$C = [(\beta_i, \beta_j^\vee)] = \begin{bmatrix} 2 & -m & 0 & \cdots & 0 \\ -m & 2 & * & \cdots & * \\ 0 & * & 2 & \cdots & * \\ \cdot & & & & \\ \cdot & & & & \\ 0 & * & * & \cdots & 2 \end{bmatrix}$$

which is rank $r+1$ of indefinite type for $m \geq 2$, where the submatrix obtained by removing the first row and column of C is the finite type Cartan matrix C_r^{fin} determined by $\alpha_1, \ldots, \alpha_r$. We also denote this subalgebra by $\mathcal{A}_{r+1}^{indef}(m)$.

Proof. As in Theorem 3.2, for $m \geq 0$, the roots β_i, $0 \leq i \leq r$, satisfy the conditions of Theorem 3.1 and therefore determine a KM subalgebra of $\mathcal{A}$ whose Cartan matrix is as shown because $(\delta, \delta) = 0$, $(\delta, \alpha_i) = 0$ for $0 \leq i \leq r$, and $(\delta, \alpha_{-1}) = -1$. Note that the submatrix obtained by removing the first row and column of C is the finite type Cartan matrix C_r^{fin} determined by $\alpha_1, \ldots, \alpha_r$. Let C_{r-1}^{fin} be the finite type Cartan matrix obtained from C_r^{fin} by deleting its first row and column. Then we see that

$$\det(C) = 2 \ \det(C_r^{fin}) - m^2 \ \det(C_{r-1}^{fin}).$$

Using the table of values of these determinants on page 53 of [**Kac**], we find that $\det(C) < 0$ in all cases for $m \geq 2$, guaranteeing that the subalgebra will be indefinite. $\square$

For example, if the finite dimensional algebra with simple roots $\alpha_1, \ldots, \alpha_r$ is of type A_r, then it is easy to see that $\det(C) = 2(r+1) - m^2 r$, which will be negative when $m^2 > 2(r+1)/r$. For $r \geq 2$, this is true for $m \geq 2$, and for $r = 1$ this is true for $m \geq 3$. The $r = 1$ case is rather extreme since we only have $\beta_0 = \alpha_{-1}$ and $\beta_1 = m\delta + \alpha_1$ giving the rank 2 Cartan matrix already studied in Theorem 3.2.

The above subalgebras $\mathcal{A}_{r+1}^{indef}(m)$ of $\mathcal{A}$ for different values of $m \geq 2$ have inequivalent Cartan matrices. Therefore, $\mathcal{A}$ possesses infinitely many nonisomorphic indefinite KM subalgebras corresponding to the infinitely many ways of slicing the forward light-cone in root space by time-like hyperplanes such that the hyperboloidal intersections become more and more steeply inclined with increasing m. For rank 3 all subalgebras were again hyperbolic, but the indefinite KM algebras obtained in this way for higher rank in general are no longer hyperbolic even if the original subalgebra is hyperbolic.

The above construction can be further modified as follows to obtain subalgebras of rank $r + 2$.

THEOREM 3.4. *Let the notation be as in Theorem 3.3. Fix any integer $m \geq 1$ and define*

$$\gamma_{-1} = \alpha_{-1}, \quad \gamma_0 = \gamma_0^{(m)} = (m-1)\delta + \alpha_0, \quad \gamma_j = \alpha_j \quad \text{for } 1 \leq j \leq r.$$

Then the KM Lie subalgebra $\mathcal{A}(\gamma_{-1}, \cdots, \gamma_r)$ constructed in Theorem 3.1 has Cartan matrix

$$C = [(\gamma_i, \gamma_j^\vee)] = \begin{bmatrix} 2 & -m & 0 & \cdots & 0 \\ -m & 2 & * & \cdots & * \\ 0 & * & 2 & \cdots & * \\ \cdot & & & & \\ \cdot & & & & \\ 0 & * & * & \cdots & 2 \end{bmatrix}$$

which is rank $r + 2$ of indefinite type for $m \geq 1$, where the submatrix obtained by removing the first row and column of C is the affine type Cartan matrix C_{r+1}^{aff} determined by $\alpha_0, \ldots, \alpha_r$. We also denote this subalgebra by $\mathcal{A}_{r+2}^{indef}(m)$. The affine subalgebra with simple roots $\gamma_0, \cdots, \gamma_r$ has minimal null root $\sum_{j=0}^r n_j \gamma_j = m\delta$.

PROOF. The proof is as in the previous theorem, but the submatrix obtained by removing the first row and column of C is the affine type Cartan matrix C_{r+1}^{aff} determined by $\alpha_0, \ldots, \alpha_r$, and C_r^{fin} is the finite type Cartan matrix obtained from C_{r+1}^{aff} by deleting its first row and column, so

$$\det(C) = 2 \ \det(C_{r+1}^{aff}) - m^2 \ \det(C_r^{fin}).$$

Since $\det(C_{r+1}^{aff}) = 0$, and $\det(C_r^{fin}) > 0$, we find that $\det(C) < 0$ in all cases for $m \geq 1$, guaranteeing that the subalgebra will be indefinite. Of course, the case when $m = 1$ just gives the original algebra $\mathcal{A}$, so the only new content is for $m \geq 2$. $\square$

For the algebra $\mathcal{F}$, and for $m \geq 2$, this procedure yields infinitely many inequivalent rank 3 subalgebras with the Cartan matrices

$$\begin{bmatrix} 2 & -m & 0 \\ -m & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

not previously known to exist inside of $\mathcal{F}$. In the next section we will explore the geometrical meaning of these subalgebras using the Weyl group of $\mathcal{F}$.

4. Indefinite subalgebras of $\mathcal{F}$ from the Weyl group

A beautiful geometrical way to find subalgebras of $\mathcal{F}$ is to examine the action of the Weyl group, W, which preserves each surface of fixed determinant. In fact, it preserves each sheet of the two sheeted hyperboloids with fixed positive integral determinant. Any one of these sheets can be taken as a model of the hyperbolic plane, isometric to the Poincaré disk, whose boundary corresponds to the light-cone. In Figure 1 we have shown the fixed points of the simple reflections which generate W, the triangular fundamental domain $\mathcal{D}$ for W, and several other reflection lines corresponding to some other positive real roots of $\mathcal{F}$ which are obtained by applying Weyl group elements to the simple roots. A complete tesselation of the disk would be obtained by including all reflection lines, which are in one-to-one correspondence with all of the positive real roots of $\mathcal{F}$, Φ^+_{real}. In that tesselation there would be a bijection between the elements of the Weyl group, W, and the images of the fundamental domain $\mathcal{D}$. In Figure 1 we see $\mathcal{D}$ is the triangle with angles $\pi/2$, $\pi/3$ and 0, and sides labelled by the simple roots. Six images of $\mathcal{D}$ form a larger triangle with all vertices on the boundary and all angles zero. The three sides of that larger triangle are labelled α_1, $r_0\alpha_1 = \alpha_1 + 2\alpha_0$ and $r_{-1}r_0\alpha_1 = \alpha_1 + 2\alpha_0 + 2\alpha_{-1}$. Since that triangle has all angles zero, and contains 6 copies of $\mathcal{D}$, it means that reflections with respect to these three roots generate a subgroup of W of index 6 isomorphic to the hyperbolic triangle group $T(\infty, \infty, \infty)$. Using Theorem 3.1, those roots determine a rank 3 hyperbolic subalgebra of $\mathcal{F}$, inequivalent to any of the algebras in the series found at the end of the last section.

More generally, for any $n \geq 1$, let $S = \{\beta_1, \cdots, \beta_n\}$ be a subset of n roots from Φ^+_{real}, corresponding to reflection lines in the Poincaré disk, such that $\beta_i - \beta_j \notin \Phi$. Then the KM subalgebra $\mathcal{F}_S = \mathcal{F}(\beta_1, \cdots, \beta_n)$ determined by Theorem 3.1 has Weyl group, W_S, generated by the reflections $r_{\beta_1}, \cdots, r_{\beta_n}$. The Cartan matrix of $\mathcal{F}_S$ is $C_S = [(\beta_i, \beta_j)]$ since all real roots of $\mathcal{F}$ are of squared length 2. The fundamental domain $\mathcal{D}_S$ of W_S is a union of images of $\mathcal{D}$, and the number of such images will be equal to the index of W_S in W. Using Figure 1 we can find choices of S which give either finite or infinite index sub-Weyl groups. Our first example was mentioned above. We leave it to the reader to check the condition that $\beta_i - \beta_j \notin \Phi$ for each choice of S.

Example 4.1. If S consists of the roots

$$\beta_1 = \alpha_1, \quad \beta_2 = r_0\alpha_1, \quad \beta_3 = r_{-1}r_0\alpha_1,$$

then $\mathcal{F}_S$ is a rank 3 subalgebra with Cartan matrix

$$\begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$

and Weyl group $W_S = T(\infty,\infty,\infty)$ of index 6 in W.

There are other hyperbolic triangles with all angles zero in Figure 1, for example, the one with sides labelled α_1, α_0 and $r_{-1}r_0r_1r_0\alpha_1 = 3\alpha_1 + 4\alpha_0 + 4\alpha_{-1}$. There are also 6 images of the fundamental triangle in this one, so these three reflections generate a subgroup of index 6 in W. The Cartan matrix generated from these three roots, taken as simple, is the same as the one above. Is there an automorphism of $\mathcal{F}$ which interchanges these two isomorphic subalgebras? Are these two index 6 subgroups of W conjugate?

Example 4.2. If S consists of the roots

$$\beta_1 = r_1r_0\alpha_1 = 3\alpha_1 + 2\alpha_0,$$

$$\beta_2 = r_0r_{-1}r_1r_0\alpha_1 = 3\alpha_1 + 6\alpha_0 + 2\alpha_{-1},$$

$$\beta_3 = r_{-1}r_0r_1r_0\alpha_1 = 3\alpha_1 + 4\alpha_0 + 4\alpha_{-1},$$

then $\mathcal{F}_S$ is a rank 3 subalgebra with Cartan matrix

$$\begin{bmatrix} 2 & -10 & -10 \\ -10 & 2 & -10 \\ -10 & -10 & 2 \end{bmatrix}$$

and Weyl group $W_S = T(\infty,\infty,\infty)$ of infinite index in W because there are no relations between the generating reflections and the area enclosed by the reflection lines contains an infinite number of copies of $\mathcal{D}$.

Example 4.3. If S consists of the roots

$$\beta_1 = \alpha_1, \qquad \beta_2 = r_0r_1r_{-1}r_0\alpha_1 = 3\alpha_1 + 6\alpha_0 + 2\alpha_{-1},$$

$$\beta_3 = r_0r_{-1}r_0r_1r_0\alpha_1 = 3\alpha_1 + 6\alpha_0 + 4\alpha_{-1},$$

then $\mathcal{F}_S$ is a rank 3 subalgebra with Cartan matrix

$$\begin{bmatrix} 2 & -6 & -6 \\ -6 & 2 & -2 \\ -6 & -2 & 2 \end{bmatrix}$$

and Weyl group $W_S = T(\infty,\infty,\infty)$ of infinite index in W because there are no relations between the generating reflections and the area enclosed by the reflection lines contains an infinite number of copies of $\mathcal{D}$.

Example 4.4. If S consists of the roots

$$\beta_1 = r_0\alpha_{-1} = \alpha_0 + \alpha_{-1}, \qquad \beta_2 = r_1\alpha_0 = 2\alpha_1 + \alpha_0,$$

$$\beta_3 = r_0r_1r_0\alpha_1 = 3\alpha_1 + 4\alpha_0,$$

then $\mathcal{F}_S$ is a rank 3 subalgebra with Cartan matrix

$$\begin{bmatrix} 2 & -3 & -2 \\ -3 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$

and Weyl group $W_S = T(\infty,\infty,\infty)$ of infinite index in W because there are no relations between the generating reflections and the area enclosed by the reflection lines contains an infinite number of copies of $\mathcal{D}$.

EXAMPLE 4.5. If S consists of the four roots

$$\begin{aligned}
\beta_1 &= r_0\alpha_{-1} &&= \alpha_0 + \alpha_{-1},\\
\beta_2 &= r_{-1}r_1r_0\alpha_1 &&= 3\alpha_1 + 2\alpha_0 + 2\alpha_{-1},\\
\beta_3 &= r_1r_0\alpha_1 &&= 3\alpha_1 + 2\alpha_0,\\
\beta_4 &= r_0r_1r_0\alpha_1 &&= 3\alpha_1 + 4\alpha_0
\end{aligned}$$

then $\mathcal{F}_S$ is a rank 3 subalgebra with Cartan matrix

$$A = [A_{ij}] = \begin{bmatrix} 2 & -2 & -4 & -2 \\ -2 & 2 & -2 & -10 \\ -4 & -2 & 2 & -2 \\ -2 & -10 & -2 & 2 \end{bmatrix}$$

and Weyl group

$$W_S = \langle r_{\beta_1}, r_{\beta_2}, r_{\beta_3}, r_{\beta_4} \mid r_{\beta_i}^2 = 1 \rangle.$$

From Figure 1 we see that there are 12 fundamental triangles enclosed by these reflecting lines, so the index of W_S in W is 12.

In $\mathcal{F}_S$, the four H_i are linearly dependent in the three dimensional space H, as are the new simple roots $\beta_1, \cdots, \beta_4$. In fact, we have $2\beta_1 - \beta_2 + 2\beta_3 - \beta_4 = 0$. It is certainly possible for a Cartan matrix to be degenerate and still define a Kac-Moody algebra, but Theorem 3.1 gives the Serre relations because for $1 \le i \ne j \le 4$, $\beta_i - \beta_j$ is not a root of $\mathcal{F}$. There are several ways to solve the dependency problem ([**Kac, Jur1, Nie**]), for example, by adjoining some derivations to the Cartan subalgebra so as to make the simple roots linearly independent, but then the resulting algebra will have Cartan subalgebra larger than the Cartan subalgebra of $\mathcal{F}$. The same considerations occur when generalizing these ideas to the higher rank algebras $\mathcal{A}$.

It would be interesting if one could use the geometry of the tesselated disk to classify the subgroups of the Weyl group of $\mathcal{F}$ and use that to classify the subalgebras coming from Theorem 3.1.

We would like to finish this section by showing how the series of subalgebras of $\mathcal{F}$ found at the end of the last section fit into the geometrical point of view given in this section. When Theorem 3.4 is applied to $\mathcal{F}$, for $m \ge 1$, the set S is

$$\gamma_{-1} = \alpha_{-1}, \quad \gamma_0 = \gamma_0^{(m)} = (m-1)\delta + \alpha_0, \quad \gamma_1 = \alpha_1$$

so the Weyl group W_S is generated by two of the simple generators of W, r_{-1} and r_1, along with one other reflection, $r_{\gamma_0^{(m)}}$. It is not hard to check that

$$\gamma_0^{(1)} = \alpha_0,\ \gamma_0^{(2)} = r_0\alpha_1,\ \gamma_0^{(3)} = r_0r_1\alpha_0,\ \gamma_0^{(4)} = r_0r_1r_0\alpha_1,\ \cdots,$$

which can be seen in Figure 1. The case of $m = 1$ just gives W, but when $m = 2$ we see that the three reflecting lines enclose a fundamental domain $\mathcal{D}_S$ containing three images of $\mathcal{D}$, and the angles of the triangle are $\pi/2$, 0 and 0, so in that case $W_S = T(2, \infty, \infty)$ is of index 3 in W. But for $m \ge 3$, the three sides do not form a triangle, and there are infinitely many copies of $\mathcal{D}$ in their fundamental domain, and the index of W_S in W is infinite. We would also like to mention that

$$\begin{bmatrix} -1 & m \\ 0 & 1 \end{bmatrix}$$

is the matrix in $PGL_2(\mathbf{Z})$ which represents the reflection $r_{\gamma_0^{(m)}}$ as discussed in section 2.

5. Borcherds algebras as subalgebras of hyperbolic KM algebras

Although one might have been surprised to find so many inequivalent indefinite KM subalgebras in any Lorentzian algebra $\mathcal{A}$, it is perhaps even more surprising to find Borcherds algebras as proper subalgebras in $\mathcal{F}$, in any Lorentzian algebra $\mathcal{A}$, and even in rank 2 hyperbolics. As is well known, Borcherds (or generalized KM) algebras can be defined in terms of a generalized Cartan matrix and a set of generators and relations just like standard KM algebras, but are distinguished by the existence of *imaginary simple roots*. These correspond to zero or negative diagonal entries in the Cartan matrix, with corresponding modifications of the Serre relations [**B1**]. (See also [**Jur1, Nie**].) Moreover, the multiplicity of a simple imaginary root may be greater than one.

To explain the basic idea let us return to Theorem 3.2. As we have seen, for each m, the hyperbolic algebra $\mathcal{H}(m)$ can be embedded into $\mathcal{F}$ by identifying its two simple roots β_0 and β_1 in the root system of $\mathcal{F}$, and the corresponding simple root generators as multiple commutators of the generators of $\mathcal{F}$. Let us also choose the $\pm$ sign to be $-$ in the definition of β_1. The root system of $\mathcal{H}(m)$ is contained in the linear subspace spanned by β_0 and β_1 inside the root lattice of $\mathcal{F}$. As we already explained, this subspace gives a hyperbolic section of the light-cone in the root lattice of $\mathcal{F}$. The root space $\mathcal{H}(m)_\beta$ associated with any root β in this hyperplane is contained in, but in general will not equal the root space $\mathcal{F}_\beta$. Rather we will have strict inequality for "sufficiently imaginary" roots β, viz.

$$\mathrm{Mult}_{\mathcal{H}(m)}(\beta) < \mathrm{Mult}_{\mathcal{F}}(\beta).$$

(Actually the difference in dimension of these two root spaces will grow exponentially as β is moved deeper into the light-cone.) As an example, let us take the rank 2 "Fibonacci" algebra $\mathcal{H}(3)$ (so called because of its connection with the Fibonacci numbers [**F**]), and in it the root $\beta = 2\beta_0 + 2\beta_1$. Then, from the table on page 214 of [**Kac**], where this root is denoted by $(2,2)$, we have

$$\mathrm{Mult}_{\mathcal{H}(3)}(\beta) = 1.$$

On the other hand, with the identification of Theorem 3.2, we have $\beta = 4\alpha_1 + 6\alpha_0 + 2\alpha_{-1}$ as a root of $\mathcal{F}$. The Weyl group reflection r_0 sends β to the root $4\alpha_1 + 4\alpha_0 + 2\alpha_{-1}$ which is denoted by $(4,4,2)$ in [**Kac**], page 215, where the multiplicity of the root is given as 7, so

$$\mathrm{Mult}_{\mathcal{F}}(\beta) = 7,$$

showing that the root space $\mathcal{F}_\beta$ contains six independent vectors not contained in $H(3)_\beta$.

We are thus led to define a new algebra inside $\mathcal{F}$, as follows. Let $\Phi(\mathcal{H}(m))$ be the set of roots of $\mathcal{F}$ which are in the plane spanned by β_0 and β_1, so they are the same as the roots of $\mathcal{H}(m)$. Let $\mathfrak{h}_m$ be the span of H_0 and H_1 from Theorems 3.1 and 3.2, that is, the Cartan subalgebra of $\mathcal{H}(m)$. Then define the subspace

$$\mathcal{G}(m) = \mathfrak{h}_m \oplus \bigoplus_{\beta \in \Phi(\mathcal{H}(m))} \mathcal{F}_\beta$$

which is a proper Lie subalgebra of $\mathcal{F}$ which contains $\mathcal{H}(m)$ properly. The only subtle point involved in checking the closure of $\mathcal{G}(m)$ under brackets is that $[\mathcal{F}_\beta, \mathcal{F}_{-\beta}] \subseteq \mathfrak{h}_m$, but this follows immediately from the invariance of the bilinear form on $\mathcal{F}$.

We can think of $\mathcal{G}(m)$ as an extension of $\mathcal{H}(m)$ in the following way. For $\beta = a_0\beta_0 + a_1\beta_1 \in \Phi(\mathcal{H}(m))^+$, $0 \le a_0, a_1 \in \mathbf{Z}$, define the $\mathcal{H}(m)$-height $\mathcal{H}t(\beta) = a_0 + a_1$. Note that this is not the same as the height of β as a root of $\mathcal{F}$. Define a sequence of extensions, $\mathcal{G}^{(i)}(m)$, recursively, beginning with $\mathcal{G}^{(1)}(m) = \mathcal{H}(m)$. For $i > 1$ let $\mathcal{G}^{(i)}(m)$ be the Lie algebra containing $\mathcal{G}^{(i-1)}(m)$ and the additional generators taken from the complement of $\mathcal{G}^{(i-1)}(m)_\beta$ in $\mathcal{F}_\beta$ for all $\beta \in \Phi(\mathcal{H}(m))^+$ with $\mathcal{H}t(\beta) = i$. This amounts to adding a new imaginary simple root for each such β with simple multiplicity [**BGGN, BGN**] (not to be confused with the root multiplicity of β)

$$\mu(\beta) = \dim(\mathcal{F}_\beta) - \dim(\mathcal{G}^{(i-1)}(m)_\beta).$$

Then, we have

$$\mathcal{G}(m) = \bigcup_{i=1}^{\infty} \mathcal{G}^{(i)}(m).$$

However, to work out the complete system of imaginary simple roots, their multiplicities, and the associated generalized Cartan matrix even for these simple examples would be an extremely difficult task. This would in particular require full knowledge of the root multiplicities of $\mathcal{F}$. Fortunately, this is not necessary because we can invoke the following theorem of Borcherds [**B2**] (also see [**Jur1**]).

THEOREM 5.1. *A Lie algebra $\mathfrak{g}$ is a Borcherds algebra if it has an almost positive definite contravariant form $\langle\ |\ \rangle$, which means that $\mathfrak{g}$ has the following properties:*

1. *(Grading)* $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ *with* $\dim \mathfrak{g}_n < \infty$ *for* $n \neq 0$*;*
2. *(Involution)* $\mathfrak{g}$ *has an involution* θ *which acts as* -1 *on* $\mathfrak{g}_0$ *and maps* $\mathfrak{g}_n$ *to* $\mathfrak{g}_{-n}$*;*
3. *(Invariance)* $\mathfrak{g}$ *carries a symmetric invariant bilinear form* $(\ |\ )$ *preserved by* θ *and such that* $(\mathfrak{g}_m|\mathfrak{g}_n) = 0$ *unless* $m + n = 0$*;*
4. *(Positivity) The contravariant form* $\langle x|y\rangle := -(\theta(x)|y)$ *is positive definite on* $\mathfrak{g}_n$ *if* $n \neq 0$*.*

Thus, we have

THEOREM 5.2. *For all $m \ge 3$, $\mathcal{G}(m)$ is a Borcherds algebra such that*

$$\mathcal{H}(m) \subset \mathcal{G}(m) \subset \mathcal{F}.$$

PROOF. All properties listed in Theorem 5.1 are satisfied for $\mathcal{G}(m)$ because they are manifestly true for $\mathcal{F}$. $\square$

In Theorem 3.4 we have seen how any Lorentzian KM algebra $\mathcal{A}$ of rank $r+2$ contains an infinite series of inequivalent indefinite KM subalgebras, $\mathcal{A}_{r+1}^{indef}(m)$, for $m \ge 2$. Let $\Phi(\mathcal{A}_{r+1}^{indef}(m))$ be the set of roots of $\mathcal{A}$ which are in the hyperplane spanned by the simple roots β_i, $0 \le i \le r$, of $\mathcal{A}_{r+1}^{indef}(m)$, and let $\mathfrak{h}_m$ be its Cartan subalgebra. Then defining

$$\mathcal{G}(\mathcal{A}_{r+1}^{indef}(m)) = \mathfrak{h}_m \oplus \bigoplus_{\beta\in\Phi(\mathcal{A}_{r+1}^{indef}(m))} \mathcal{A}_\beta$$

gives a proper Lie subalgebra of $\mathcal{A}$ generalizing the previous construction.

THEOREM 5.3. *For all $m \ge 2$, $\mathcal{G}(\mathcal{A}_{r+1}^{indef}(m))$ is a Borcherds algebra such that*

$$\mathcal{A}_{r+1}^{indef}(m) \subset \mathcal{G}(\mathcal{A}_{r+1}^{indef}(m)) \subset \mathcal{A}.$$

Let's denote more briefly by $\mathcal{G}$ any of the "hyperplane Borcherds subalgebras" just constructed inside $\mathcal{A}$, and let $\mathcal{H}$ denote the indefinite KM subalgebra properly contained in $\mathcal{G}$. Then we have the following decomposition

$$\mathcal{G} = \mathcal{M}_- \oplus \mathcal{H} \oplus \mathcal{M}_+$$

where $\mathcal{M}_+$ and $\mathcal{M}_-$ are supported on positive and negative roots, respectively. This decomposition corresponds to a similar decomposition found in [**Jur2**] for all Borcherds algebras,

$$\tilde{\mathcal{G}} = \mathcal{M}_- \oplus (\mathcal{H} \oplus \tilde{\mathfrak{h}}) \oplus \mathcal{M}_+,$$

where $\tilde{\mathfrak{h}}$ is an infinite dimensional extension of the Cartan subalgebra of the KM algebra $\mathcal{H}$ which makes all the imaginary simple roots linearly independent. This extension is analogous to the extension of the Cartan subalgebra mentioned at the end of the previous section. But this extension would not be contained in $\mathcal{A}$. As shown in [**Jur2**], $\mathcal{M}_+$ and $\mathcal{M}_-$ are free Lie algebras. It would be interesting to determine their structure as $\mathcal{H}$-modules. Similar structures were studied in [**BGGN**].

Finally, we would like to note that there are Borcherds algebras inside the rank 2 hyperbolics, and in particular, in $\mathcal{H}(3)$, whose positive roots are shown in Figure 2. Note that the positive real roots are shown by open circles and the positive imaginary roots by solid dots. The figure also shows the simple reflection lines and root multiplicities. We draw the reader's attention to the central vertical line in the figure, and define the subalgebra $\mathcal{G}$ which is the direct sum of all the root spaces along that line, including the negative root spaces not shown and the one-dimensional subspace of the Cartan subalgebra spanned by $h_1 + h_2$. This is the simplest example of a Borcherds algebra embedded inside a hyperbolic KM algebra. In this case we have the decomposition

$$\mathcal{G} = \mathcal{M}_- \oplus sl_2 \oplus \mathcal{M}_+$$

and it would not be hard to determine the number of free generators in the root spaces of $\mathcal{M}_+$ by using the formulas in [**Jur2**] for the dimensions of graded subspaces in free Lie algebras with graded generators.

6. Appendices

Figure 1: Poincaré Disk Model of Hyperbolic Plane Tesselated
By the Hyperbolic Triangle Group $T(2, 3, \infty)$

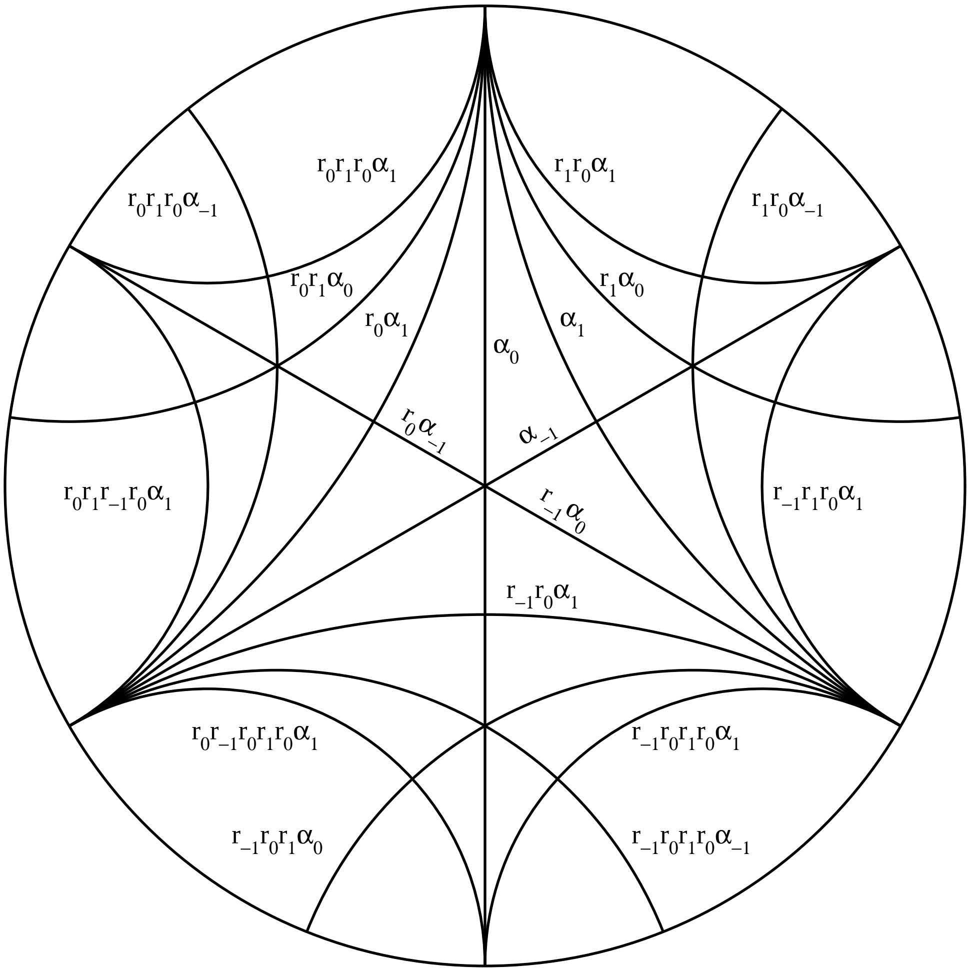
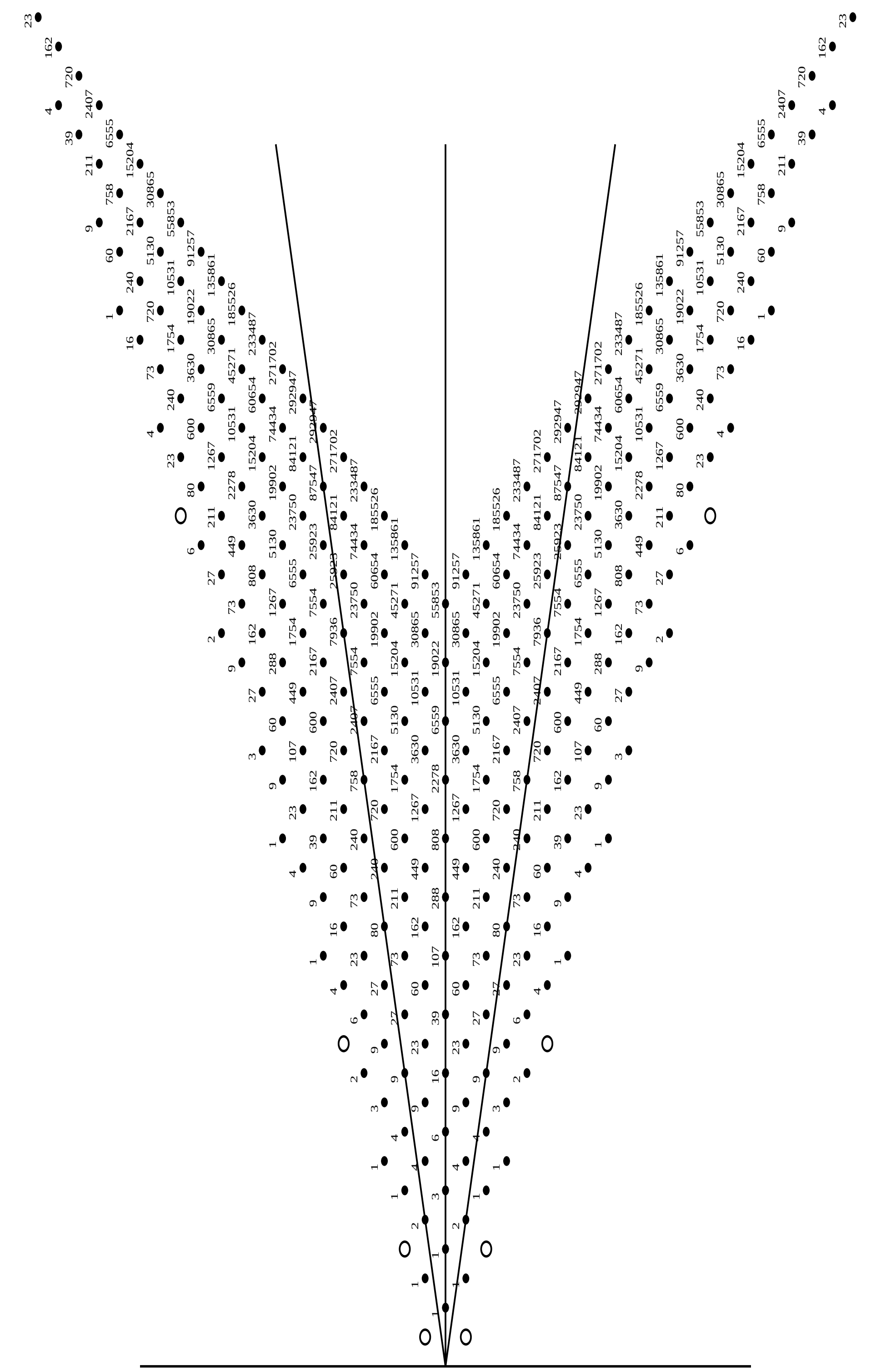


Figure 2: Hyperbolic Root System For The Fibonacci Algebra $\mathcal{H}(3)$



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