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A Hyperbolic Kac-Moody Algebra and the Theory of Siegel Modular Forms of Genus 2

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Dedicated to Nathan Jacobson

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1. Introduction

In the last few years the theory of Kac-Moody Lie algebras has drawn considerable attention because of its numerous connections with other topics in mathematics and physics. Most of these results concern only the simplest class of infinite dimensional Kac-Moody algebras known as the affine Lie algebras. Except for some general results and the papers [4, 24, 35], little is known about the hyperbolic Lie algebras, the next class of Kac-Moody algebras after the affines. In this paper we restrict ourselves to a detailed investigation of one particular hyperbolic Lie algebra, $\mathfrak{F}$, although our techniques certainly generalize. Our main results are a construction of $\mathfrak{F}$ and the relationship between the representation theory of $\mathfrak{F}$ and the theory of Siegel modular forms of genus 2. One of the purposes of this paper is to show the richness of the algebraic structures associated with the hyperbolic Kac-Moody algebras and their newly discovered connections with classical mathematics. These lead us to expect even more important discoveries in this direction.

The definition of a Kac-Moody algebra by generators and relations (see Sect. 2) is based on an integral square matrix A, called the Cartan matrix. We will

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study the case where the Cartan matrix is

$$A=(A_{ij})=\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{1.1}$$

and the corresponding Dynkin diagram is

 (1.2)

The Cartan matrix A determines a symmetric bilinear form $(\cdot,\cdot)$ on $\mathbb{R}^3=\mathbb{R}\alpha_1+\mathbb{R}\alpha_2+\mathbb{R}\alpha_3$ defined by $(\alpha_i,\alpha_j)=A_{ij}$ for $i,j\in\{1,2,3\}$. It also determines a discrete group W, called the Weyl group, which is generated by reflections with respect to α_i for $i=1,2,3$. The connections with classical mathematics start from the fact that $W\approx \mathrm{PGL}_2(\mathbb{Z})$, one of the most extensively investigated discrete groups. The root lattice

$$Q=\mathbb{Z}\alpha_1+\mathbb{Z}\alpha_2+\mathbb{Z}\alpha_3 \tag{1.3}$$

and the weight lattice

$$P=\{\lambda\in\mathbb{R}^3\,|\,(\lambda,\alpha_i)\in\mathbb{Z} \text{ for } i=1,2,3\} \tag{1.4}$$

also have nice realizations,

$$Q=\left\{\begin{pmatrix} a & b \\ b & c \end{pmatrix}\middle|\, a,b,c\in\mathbb{Z}\right\} \tag{1.5}$$

and

$$P=\left\{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\middle|\, a,b,c\in\mathbb{Z}\right\}, \tag{1.6}$$

so that W acts in the usual way

$$w\cdot\lambda=w\lambda w^t \tag{1.7}$$

for $w\in W$ and $\lambda\in P$. The classification of W-orbits in P can then be seen to be equivalent to the classification of equivalence classes of integral binary quadratic forms

$$F(x,y)=ax^2+bxy+cy^2\,. \tag{1.8}$$

In the case of positive semidefinite forms the classification is given in terms of the dominant integral weights P^{++} in Sect. 2.

One of the most important results in the theory of affine Lie algebras is their explicit realization which gives a root multiplicity formula and clarifies their structure. In Sect. 4 we give a construction of the hyperbolic algebra $\mathfrak{F}$ which can be easily generalized. Let us denote by $\mathfrak{F}_0$ the affine Lie subalgebra of $\mathfrak{F}$ corresponding to the Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Let V be a basic representation of $\mathfrak{F}_0$ (see [10, 17]) and let V^* be the dual (contragredient) representation. Let $\mathfrak{F}_0^e$ be a one-dimensional extension of $\mathfrak{F}_0$ obtained by adjoining a certain derivation (see Sect. 3) and let $\{x_i\,|\,i\in I\}$ be an orthonormal basis of $\mathfrak{F}_0^e$ with respect to the invariant symmetric bilinear form $(\cdot,\cdot)$ normalized so that $(\alpha_i, \alpha_i)=2$ for $i=1,2$. Then we define a $\mathbb{Z}$-graded Lie algebra

$$\mathfrak{G} = \ldots + [V^*, V^*] + V^* + \mathfrak{F}_0^e + V + [V, V] + \ldots, \tag{1.9}$$

where for $x \in \mathfrak{F}_0^e$, $v \in V$, $v^* \in V^*$ we have

$$[x, v] = x \cdot v, \quad [x, v^*] = x \cdot v^* \tag{1.10}$$

and

$$[v^*, v] = -\sum_{i \in I} (v^*, x_i \cdot v) x_i. \tag{1.11}$$

This algebra has a unique maximal graded ideal

$$\mathfrak{I} = \sum_{n \in \mathbb{Z}} \mathfrak{I}_n \tag{1.12}$$

and the hyperbolic algebra $\mathfrak{F}$ is isomorphic to $\mathfrak{G}/\mathfrak{I}$.

A description of the ideal $\mathfrak{I}$ allows us to reduce the problem of determining root multiplicities of $\mathfrak{F}$ to the study of weight multiplicities of some representations of $\mathfrak{F}_0^e$. The weight multiplicities of $\mathfrak{F}_0^e$ are all one, and those of V are known [6, 16] to be values of the classical partition function, and in fact we have

$$\operatorname{Mult}\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} = p(n+1), \tag{1.13}$$

where,

$$\sum_{n \geqq 0} p(n) q^n = \prod_{n \geqq 1} (1-q^n)^{-1}. \tag{1.14}$$

It can be seen that $\mathfrak{I}_{-1}$, $\mathfrak{I}_0$, $\mathfrak{I}_1$ are trivial, so we have complete knowledge of the root multiplicities from those graded pieces. We are able to compute the root multiplicities from $[V, V]/\mathfrak{I}_2$ as follows. We can show that the decomposition of the tensor product

$$V \otimes V = S(V) + A(V) \tag{1.15}$$

into symmetric and antisymmetric tensors corresponds to the decomposition of $V \otimes V$ into two "strings" of equivalent $\mathfrak{F}_0$-modules [see (3.42)]. The proof of this result uses the techniques of vertex operators studied in [8, 10]. We also show that $\mathfrak{I}_2 \subset A(V) \approx [V, V]$ is exactly one irreducible standard $\mathfrak{F}_0^e$-module, and using the outer multiplicity formula for $V \otimes V$ from [5] we obtain the following result. We have

$$\operatorname{Mult}\begin{pmatrix} n & 0 \\ 0 & 2 \end{pmatrix} = p'(2n+1) \tag{1.16}$$

and

$$\operatorname{Mult}\begin{pmatrix} n & 1 \\ 1 & 2 \end{pmatrix} = p'(2n), \tag{1.17}$$

where $p'(n)$ is a modified partition function

$$\sum_{n \geqq 0} p'(n) t^n = \left[\prod_{n \geqq 1} (1-t^n)^{-1}\right](1 - t^{20} + t^{22} - t^{24} + \ldots) \tag{1.18}$$

[see (4.36)]. It is most remarkable that the first nonzero term beyond 1 in the second factor on the right side of (1.18) ist t^{20}, so that $p'(n)=p(n)$ for $0 \leqq n \leqq 19$ but $p'(20) \neq p(20)$. Theoretically our construction of $\mathfrak{F}$ should allow us to find all root multiplicities by studying the weight multiplicities of certain $\mathfrak{F}_0^e$-submodules in tensor powers of V. We do not, however, have a closed formula for the multiplicity of an arbitrary root of $\mathfrak{F}$ other than the general formula given in [36].

A complete knowledge of the root multiplicities of $\mathfrak{F}$ would yield a quite remarkable identity which comes from the Weyl-Kac denominator formula for $\mathfrak{F}$. We recall first that this formula for $\mathfrak{F}_0^e$ is the classical Jacobi triple product identity which may be written as

$$\begin{aligned} J_0(z,\tau) &= \sum_{j\in\mathbb{Z}} (-1)^j e^{4\pi i(j+1/2)z} q^{(j+1/2)^2} \\ &= 2i\sin(2\pi z) \prod_{n\geqq 1} (1-q^{2n})(1-q^{2n}e^{4\pi i z})(1-q^{2n}e^{-4\pi i z}), \end{aligned} \tag{1.19}$$

where $q=e^{\pi i\tau}$ for $\mathrm{Im}(\tau)>0$ and $z\in\mathbb{C}$. The denominator formula for $\mathfrak{F}$ is

$$\begin{aligned} &\sum_{g\in PGL_2(\mathbb{Z})} \det(g) e^{2\pi i \operatorname{Tr}((gPg^t - P)\mathfrak{Z})} \\ &= \prod_{0 \leq N \in S_2(\mathbb{Z})} (1-e^{2\pi i \operatorname{Tr}(N\mathfrak{Z})})^{\mathrm{Mult}(N)} \prod_{\substack{\det(N)=-1 \\ N\in v(R^-)}} (1-e^{2\pi i \operatorname{Tr}(N\mathfrak{Z})}), \end{aligned} \tag{1.20}$$

where $P=\begin{pmatrix} 3 & 1/2 \\ 1/2 & 2 \end{pmatrix}$, $\mathfrak{Z}=\begin{pmatrix} z_3 & z_1 \\ z_1 & z_2 \end{pmatrix}$, $S_2(\mathbb{Z})$ is the set of symmetric integral 2×2 matrices and $v(R^-)$ is the set of matrices in $S_2(\mathbb{Z})$ corresponding to the negative roots of $\mathfrak{F}$ (see Theorem 4.10). Undoubtedly this identity should have a direct analytic proof analogous to that of the famous Jacobi identity. We believe that such a proof would shed new light on the root multiplicities, $\mathrm{Mult}(N)$, perhaps showing that they have a number-theoretical meaning connected with ideal classes of imaginary quadratic fields.

The denominator $J_0(z,\tau)$ for $\mathfrak{F}_0^e$, trivially related to the classical Jacobi theta function

$$\Theta_1(z,\tau) = -i \sum_{n\in\mathbb{Z}} (-1)^n e^{2\pi i(n+1/2)z} q^{(n+1/2)^2} \tag{1.21}$$

by the formula

$$J_0(z,\tau) = i\Theta_1(2z,\tau), \tag{1.22}$$

has another remarkable property besides the product expansion (1.19). If we define

$$J(z,t,\tau) = J_0(z,\tau) e^{4\pi i t} \tag{1.23}$$

then we have the following simple transformation formula for J under the involution $\tau \to \frac{-1}{\tau}$,

$$J(z,t,\tau) = c\tau^{-1/2} J\left(\frac{-z}{\tau}, t-\frac{z^2}{\tau}, \frac{-1}{\tau}\right), \tag{1.24}$$

where c is a certain root of unity (see Corollary 5.2). Together with a transformation formula for J under the translation $\tau \to \tau+1$, this gives a formula for any transformation $\tau \to \frac{a\tau+b}{c\tau+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z})$. It is an open interesting problem to determine the transformation properties of the denominator (1.20) of $\mathfrak{F}$ under the symplectic involution $\mathfrak{Z} \to -\mathfrak{Z}^{-1}$. We note that together with the group $W \ltimes Q \approx PGL_2(\mathbb{Z}) \ltimes \mathbb{Z}^3$ this involution generates the symplectic group $Sp_2(\mathbb{Z})$, and the invariance of (1.20) under $W \ltimes Q$ is apparent. The role of these involutions is not well understood from the Lie-theoretical point of view, although some progress in this direction has been made. Kac and Peterson in [18] indicate that the characters of standard irreducible $\mathfrak{F}_0^e$-modules are invariant with respect to certain congruence subgroups of $PSL_2(\mathbb{Z})$. In Sect. 5 we study the space $\mathcal{M}_k$ of weight k $PSL_2(\mathbb{Z})$-invariant $\mathfrak{F}_0^e$-characters, and its natural subspace $\mathcal{M}_k'$. We say that an $\mathfrak{F}_0^e$-module (resp., character) is $\mathfrak{F}$-dominant if its decomposition into standard irreducible modules (resp., characters) consists only of those with highest weights which are $\mathfrak{F}$-dominant. The space $\mathcal{M}_k'$ consists of the $\mathfrak{F}$-dominant characters from $\mathcal{M}_k$. We show in Sect. 7 an isomorphism between $\mathcal{M}_k'$ and the space $\mathfrak{M}_k^2$ of genus 2 Siegel modular forms of weight k, whose dimension was found by Igusa [13] to be

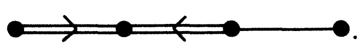
$$\dim(\mathfrak{M}_k^2) = \text{Cardinality}\,\{(a,b,c,d) \in \mathbb{Z}_+^4 \mid k = 4a+6b+10c+12d\}\,. \tag{1.25}$$

This fact implies that parallel to the correspondence between $\mathfrak{F}$-dominant standard $\mathfrak{F}_0^e$-modules and standard $\mathfrak{F}$-modules one has a correspondence between $\mathfrak{F}$-dominant $\mathfrak{F}_0^e$-characters which are $PSL_2(\mathbb{Z})$-invariant and $\mathfrak{F}$-characters which are $Sp_2(\mathbb{Z})$-invariant. The first correspondence is canonical, two modules being in correspondence if they have the same dominant integral highest weight, so one would expect the second correspondence to also be canonical. Such a canonical correspondence was known for functions of level 0 in the theory of Eisenstein series, and for functions of level 1 was recently studied by Maass [30–32] and others. In Sect. 7 we recall the Maass correspondence and give its generalization to higher levels.

The Maass space, which is a subspace of $\mathfrak{M}_k^2$, recently attracted considerable attention in the theory of Siegel modular forms because of its connection with the Saito-Kurokawa conjecture [2, 20, 22, 38, 43]. Our Lie-theoretical approach allows us to generalize the Maass correspondence to level 1 characters of any affine Lie algebra $\hat{\mathfrak{g}}$ and a specifically constructed Kac-Moody algebra $\hat{\hat{\mathfrak{g}}}$ containing $\hat{\mathfrak{g}}$. We wish to mention one example of this generalization, where the affine algebra $\hat{\mathfrak{g}}$ is of type $C_2^{(1)}$ in the notation of Kac [16] having Dynkin diagram

(1.26)

and where the hyperbolic algebra $\hat{\hat{\mathfrak{g}}}$ has Dynkin diagram



(1.27)

The Weyl group of $\hat{\hat{\mathfrak{g}}}$ is isomorphic to the Klein-Fricke group Ψ_1^* which contains as a subgroup of index 4 the Picard group $PSL_2(\mathbb{Z}[i])$. The semidirect product of the Weyl group with the root lattice, extended by the symplectic involution, generates $Sp_2(\mathbb{Z}[i])$. The lifting of $\hat{\mathfrak{g}}^e$-characters on level 1 which are $PSL_2(\mathbb{Z}[i])$-invariant by using Hecke operators in analogy with the work of Maass provides a construction of $Sp_2(\mathbb{Z}[i])$-invariant Hermitian modular forms of genus 2. This has already been noticed by Kojima in [21]. Generalizations of these results will appear in subsequent publications.

We have recently received from Yoshida a preprint [42] in which he has worked out the straightforward classification of the finite number of rank 3 hyperbolic symmetrizable Kac-Moody algebras, and noticed that the Weyl groups of these algebras are all hyperbolic triangle groups. We had known this for some time and mentioned this observation in our talks at the 789th regional meeting of the American Mathematical Society. Yoshida has also seen that the semidirect product of the Weyl group and the root lattice will be isomorphic to a discrete subgroup of a parabolic subgroup of $Sp_2(\mathbb{R})$.

2. The Kac-Moody Algebra $\mathfrak{F}$

We will study the rank 3 hyperbolic Kac-Moody Lie algebra $\mathfrak{F}$ whose Cartan matrix is

$$A=(A_{ij})=\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{2.1}$$

We recall (see [14] or [12]) that $\mathfrak{F}$ is a Lie algebra with 9 generators e_i, f_i, h_i, $i=1,2,3$ and the following relations:

$$[h_i,h_j]=0,\ [e_i,f_j]=\delta_{ij}h_i,\ [h_i,e_j]=A_{ij}e_j,\ [h_i,f_j]=-A_{ij}f_j$$

for all $i,j\in\{1,2,3\}$, and (2.2)

$$(\operatorname{ad}e_i)^{-A_{ij}+1}e_j=0=(\operatorname{ad}f_i)^{-A_{ij}+1}f_j \quad \text{for} \quad i\neq j.$$

The subalgebra $\mathfrak{h}=\mathbb{C}h_1\oplus\mathbb{C}h_2\oplus\mathbb{C}h_3$ is called a Cartan subalgebra of $\mathfrak{F}$. We denote by α_i, $i=1,2,3$, the elements in the dual space $\mathfrak{h}^*$ defined by

$$\alpha_i(h_j)=A_{ij} \quad \text{for} \quad i,j\in\{1,2,3\}. \tag{2.3}$$

The algebra $\mathfrak{F}$ decomposes into eigenspaces under the simultaneous adjoint action of $\mathfrak{h}$. These eigenspaces of $\mathfrak{F}$ are called root spaces and correspond to certain elements of $\mathfrak{h}^*$ as follows. For $\alpha\in\mathfrak{h}^*$ define

$$\mathfrak{F}^\alpha=\{x\in\mathfrak{F}\mid [h,x]=\alpha(h)x, \text{ for all } h\in\mathfrak{h}\} \tag{2.4}$$

to be the α-root space of $\mathfrak{F}$. Clearly $\mathfrak{F}^0=\mathfrak{h}$. Define the root system of $\mathfrak{F}$ to be

$$R=\{0\neq\alpha\in\mathfrak{h}^*\mid \mathfrak{F}^\alpha\neq 0\}. \tag{2.5}$$

Clearly $\mathfrak{F}^{\alpha_i}=\mathbb{C}e_i$, $\mathfrak{F}^{-\alpha_i}=\mathbb{C}f_i$ and $\mathfrak{F}=\mathfrak{h}+\sum_{\alpha\in R}\mathfrak{F}^\alpha$. It is known that any α in R can be written as an integral linear combination of α_1, α_2, α_3 with all coefficients either nonnegative or nonpositive. In the first case we say that α is a positive root and we denote these by R^+. Then $R^-=-R^+$ and $R=R^+\cup R^-$.

It is known that $\mathfrak{F}$ is simple [11] and has an invariant bilinear form $(\cdot,\cdot)$, i.e. for any $x,y,z\in\mathfrak{F}$, $([x,y],z)=(x,[y,z])$. This form is uniquely determined up to a scalar multiple. We choose this multiple so that $(h_i, h_i)=2$ for $i=1,2,3$. The set $\{\alpha_1, \alpha_2, \alpha_3\}$ of simple roots is a basis for $\mathfrak{h}^*$. Another canonical basis of $\mathfrak{h}^*$ consists of the fundamental weights ω_1, ω_2, ω_3 defined by

$$\omega_i(h_j)=\delta_{ij} \quad \text{for} \quad i,j\in\{1,2,3\}. \tag{2.6}$$

We define the root lattice of $\mathfrak{F}$ to be

$$Q=\mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \tag{2.7}$$

and the weight lattice of $\mathfrak{F}$ to be

$$P=\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \oplus \mathbb{Z}\omega_3. \tag{2.8}$$

We also define the sets

$$Q^+=\mathbb{Z}^+\alpha_1 \oplus \mathbb{Z}^+\alpha_2 \oplus \mathbb{Z}^+\alpha_3, \tag{2.9}$$

$$Q^-=-Q^+, \tag{2.10}$$

and

$$P^{++}=\mathbb{Z}^+\omega_1 \oplus \mathbb{Z}^+\omega_2 \oplus \mathbb{Z}^+\omega_3. \tag{2.11}$$

P^{++} is called the set of dominant weights of $\mathfrak{F}$.

We identify $\mathfrak{h}$ with $\mathfrak{h}^*$ by the isomorphism η determined by

$$\eta(h_i)=\alpha_i \quad \text{for} \quad i=1,2,3. \tag{2.12}$$

Thus, $\mathfrak{h}^*$ has a bilinear form, which we continue to denote by $(\cdot,\cdot)$.

We prefer to use the following basis for $\mathfrak{h}^*$;

$$\gamma_1^*=\alpha_1/2, \quad \gamma_2^*=-\alpha_1-\alpha_2-\alpha_3, \quad \gamma_3^*=-\alpha_1-\alpha_2. \tag{2.13}$$

For this basis the matrix of the form is

$$((\gamma_i^*, \gamma_j^*))=\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{2.14}$$

We have

$$\omega_1=\gamma_1^*+\gamma_2^*+\gamma_3^*, \quad \omega_2=\gamma_2^*+\gamma_3^*, \quad \omega_3=\gamma_3^*, \tag{2.15}$$

and

$$\alpha_1=2\gamma_1^*, \quad \alpha_2=-2\gamma_1^*-\gamma_3^*, \quad \alpha_3=\gamma_3^*-\gamma_2^*. \tag{2.16}$$

We also see that

$$Q=\{2n_1\gamma_1^*+n_2\gamma_2^*+n_3\gamma_3^* \mid n_1, n_2, n_3\in\mathbb{Z}\}, \tag{2.17}$$

$$Q^+=\{2n_1\gamma_1^*+n_2\gamma_2^*+n_3\gamma_3^*\in Q \mid n_1 \geqq n_2+n_3, n_2+n_3 \leqq 0, n_2 \leqq 0\}, \tag{2.18}$$

$$P=\{n_1\gamma_1^*+n_2\gamma_2^*+n_3\gamma_3^* \mid n_1, n_2, n_3\in\mathbb{Z}\}, \tag{2.19}$$

and

$$P^{++}=\{n_1\gamma_1^*+n_2\gamma_2^*+n_3\gamma_3^*\in P \mid n_3 \geqq n_2 \geqq n_1 \geqq 0\}. \tag{2.20}$$

The Weyl group W acting on $\mathfrak{h}^*$ is generated by r_1, r_2, r_3 whose action on $\mathfrak{h}^*$ is given by

$$r_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i. \tag{2.21}$$

It is known that W is a Coxeter group with relations

$$r_1^2 = r_2^2 = r_3^2 = (r_1 r_3)^2 = (r_2 r_3)^3 = 1. \tag{2.22}$$

Therefore, W is isomorphic to the extended modular group (see [34, p. 111]), which we think of as $\mathrm{PGL}_2(\mathbb{Z})$. The even subgroup $W^+ \subset W$ generated by $r_2 r_1$ and $r_1 r_3$ is then isomorphic to the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. An explicit description of the isomorphism will be given below.

The set of real roots is defined to be

$$R_W = W \cdot \{\alpha_1, \alpha_2, \alpha_3\} \tag{2.23}$$

and the set of imaginary roots is then

$$R_I = \{\alpha \in R \mid \alpha \notin R_W\}. \tag{2.24}$$

We also have positive (and negative) real and imaginary roots,

$$R_W^\pm = R_W \cap R^\pm, \quad R_I^\pm = R_I \cap R^\pm, \tag{2.25}$$

and

$$R^\pm = R \cap Q^\pm, \quad R_W^\pm = R_W \cap Q^\pm, \quad R_I^\pm = R_I \cap Q^\pm. \tag{2.26}$$

Moody has shown [35] that for hyperbolic algebras

$$R_I = \{\alpha \in Q \mid (\alpha, \alpha) \leqq 0\}. \tag{2.27}$$

With respect to the basis (2.13) the generators of W act as follows:

$$\begin{aligned} r_1(z_1\gamma_1^* + z_2\gamma_2^* + z_3\gamma_3^*) &= -z_1\gamma_1^* + z_2\gamma_2^* + z_3\gamma_3^*, \\ r_2(z_1\gamma_1^* + z_2\gamma_2^* + z_3\gamma_3^*) &= (-z_1 + 2z_2)\gamma_1^* + z_2\gamma_2^* + (-z_1 + z_2 + z_3)\gamma_3^*, \\ r_3(z_1\gamma_1^* + z_2\gamma_2^* + z_3\gamma_3^*) &= z_1\gamma_1^* + z_3\gamma_2^* + z_2\gamma_3^*. \end{aligned} \tag{2.28}$$

The elements in P, Q, and W can be realized as 2×2 matrices. Let $S_2(\mathbb{C})$ denote the complex symmetric 2×2 matrices and let $S_2(\mathbb{Z})$ denote the integral symmetric 2×2 matrices. We will also use

$$S_2'(\mathbb{Z}) = \left\{ \begin{pmatrix} n_2 & n_1/2 \\ n_1/2 & n_3 \end{pmatrix} \middle| \, n_1, n_2, n_3 \in \mathbb{Z} \right\} \supset S_2(\mathbb{Z}). \tag{2.29}$$

Each of these sets is invariant under the following action of $\mathrm{PGL}_2(\mathbb{Z})$;

$$g \cdot x = g x g^t \quad \text{for} \quad g \in \mathrm{PGL}_2(\mathbb{Z}). \tag{2.30}$$

Define the map $\nu : \mathfrak{h}^* \to S_2(\mathbb{C})$ by

$$\nu(z_1\gamma_1^* + z_2\gamma_2^* + z_3\gamma_3^*) = \begin{pmatrix} z_3 & z_1/2 \\ z_1/2 & z_2 \end{pmatrix}. \tag{2.31}$$

We also define a group homomorphism $\bar{\nu}: W \to \mathrm{PGL}_2(\mathbb{Z})$ which is uniquely determined by $\bar{\nu}(r_i) = \varrho_i$ for $i = 1, 2, 3$ where

$$\varrho_1 = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varrho_2 = \pm\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \varrho_3 = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.32}$$

are generators of $\mathrm{PGL}_2(\mathbb{Z})$. Also, $\bar{\nu}: W^+ \to \mathrm{PSL}_2(\mathbb{Z})$.

Proposition 2.1. (a) *The map* ν *defines a vector space isomorphism* $\mathfrak{h}^* \approx S_2(\mathbb{C})$ *and lattice isomorphisms* $P \approx S_2'(\mathbb{Z})$ *and* $Q \approx S_2(\mathbb{Z})$ *so that* $S_2(\mathbb{C})$ *has the bilinear form* $(\cdot, \cdot)$,

(b) *the map* $\bar{\nu}$ *gives group isomorphisms*

$$PGL_2(\mathbb{Z}) \approx W \quad \textit{and} \quad \mathrm{PSL}_2(\mathbb{Z}) \approx W^+,$$

(c) *for any* $\lambda \in \mathfrak{h}^*$, $w \in W$ *we have*

$$\nu(w \cdot \lambda) = \bar{\nu}(w) \cdot \nu(\lambda), \tag{2.33}$$

(d) *the form* $(\cdot, \cdot)$ *on* $S_2(\mathbb{C})$ *is invariant under* $\mathrm{PGL}_2(\mathbb{Z})$,

(e)

$$\nu(R_I) = \{x \in S_2(\mathbb{Z}) \mid \det x \geqq 0\},$$

$$\nu(R_I^-) = \left\{\begin{pmatrix} n_3 & n_1 \\ n_1 & n_2 \end{pmatrix} \in S_2(\mathbb{Z}) \mid n_2 n_3 \geqq n_1^2, n_2 \geqq 0, n_3 \geqq 0\right\},$$

$$\nu(R_W) = \{x \in S_2(\mathbb{Z}) \mid \det x = -1\}.$$

Proof. Everything is straightforward except for the fact that $\{x \in S_2(\mathbb{Z}) \mid \det x = -1\} \subseteqq \nu(R_W)$. Note first that $r_2 r_3(\alpha_2) = \alpha_3$, so actually we have from (2.23)

$$\nu(R_W) = \mathrm{PGL}_2(\mathbb{Z}) \cdot \{\nu(\alpha_1), \nu(\alpha_3)\}. \tag{2.34}$$

Any element $g \in \mathrm{PGL}_2(\mathbb{Z})$ may be uniquely written as

$$g = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} \frac{\alpha+\beta}{2} & \frac{\beta-\alpha}{2} \\ \frac{\gamma+\delta}{2} & \frac{\delta-\gamma}{2} \end{pmatrix}, \tag{2.35}$$

where $a, b, c, d \in \mathbb{Z}$, $ad - bc = \pm 1$, $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, $\alpha\delta - \beta\gamma = \pm 2$, $\alpha \equiv \beta (\mathrm{mod}\ 2)$ and $\gamma \equiv \delta (\mathrm{mod}\ 2)$. Also note that

$$\begin{aligned} g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g^t &= \begin{pmatrix} 2ab & ad+bc \\ ad+bc & 2cd \end{pmatrix}, \\ g\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^t &= \begin{pmatrix} a^2-b^2 & ac-bd \\ ac-bd & c^2-d^2 \end{pmatrix} = \begin{pmatrix} \alpha\beta & (\alpha\delta+\beta\gamma)/2 \\ (\alpha\delta+\beta\gamma)/2 & \gamma\delta \end{pmatrix}. \end{aligned} \tag{2.36}$$

From (2.34) it is clear that we have

$$\nu(R_W) = \left\{g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g^t \mid g \in \mathrm{PGL}_2(\mathbb{Z})\right\} \cup \left\{g\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^t \mid g \in \mathrm{PGL}_2(\mathbb{Z})\right\}. \tag{2.37}$$

Let $N=\begin{pmatrix} n_1 & n \\ n & n_2 \end{pmatrix} \in S_2(\mathbb{Z})$ have $\det(N)=-1$. One can easily find $g_1, g_2 \in \mathrm{PGL}_2(\mathbb{Z})$ such that if $n_1, n_2 \in 2\mathbb{Z}$, $N=g_1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_1^t$, and if $n_1 \notin 2\mathbb{Z}$ or $n_2 \notin 2\mathbb{Z}$, $N=g_2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g_2^t$.

Definition. Let $R_W^i=\{gv(\alpha_i)g^t \mid g\in \mathrm{PGL}_2(\mathbb{Z})\}$ for $i=1,2,3$. Then we have

$$R_W^1=\left\{g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g^t \mid g\in \mathrm{PGL}_2(\mathbb{Z})\right\} \quad \text{and} \quad R_W^2=R_W^3=\left\{g\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^t \mid g\in \mathrm{PGL}_2(\mathbb{Z})\right\}.$$

Proposition 2.2. *Let $N\in R_W^i$, $i=1$ or 3, then N has exactly one presentation in the form $N=gv(\alpha_i)g^t$ for $\det(g)=+1$ and exactly one presentation in the form $N=gv(\alpha_i)g^t$ for $\det(g)=-1$.*

Proof. From (2.36) one may see that if $gv(\alpha_i)g^t=v(\alpha_i)$ then either $g=1$ or $g=v(\alpha_i)$. So from $N=g_1v(\alpha_i)g_1^t=g_2v(\alpha_i)g_2^t$ one has $g_2^{-1}g_1=1$ or $g_2^{-1}g_1=v(\alpha_i)$. Either $g_1=g_2$ or else $g_1=g_2v(\alpha_i)$.

Definition. Let

$$R_I^0=\left\{g\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} g^t \mid 0\neq m\in\mathbb{Z}, g\in \mathrm{PGL}_2(\mathbb{Z})\right\}.$$

Proposition 2.3. *There is a one-to-one correspondence between the set $(\mathbb{Q}\cup\infty)\times(\mathbb{Z}-\{0\})$ and R_I^0.*

Proof. We have that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}=\begin{pmatrix} a^2m & acm \\ acm & c^2m \end{pmatrix},$$

where $(a,c)=1$, so that

$$R_I^0=\left\{\begin{pmatrix} a^2m & acm \\ acm & c^2m \end{pmatrix} \middle| (a,c)=1, 0\neq m\in\mathbb{Z}\right\}.$$

But $\mathbb{Q}\cup\infty$ is in one-to-one correspondence with $\{(a;c)\in\mathbb{Z}\times\mathbb{Z} \mid (a,c)=1\}$ modulo the relation $(a;c)=(-a;-c)$. The proposition is now clear.

Definition. Let $\bar{\mathbb{Q}}=\mathbb{Q}\cup\infty$. For each $q\in\bar{\mathbb{Q}}$ let

$$R_I^0(q)=\left\{\begin{pmatrix} a^2m & acm \\ acm & c^2m \end{pmatrix} \middle| a,c\in\mathbb{Z}, (a,c)=1, q=a/c, 0\neq m\in\mathbb{Z}\right\}$$

and let

$$R_W(q)=\left\{N=\begin{pmatrix} n_1 & n \\ n & n_2 \end{pmatrix}\in R_W \mid n_1c^2+n_2a^2=2nac \text{ for } q=a/c, (a,c)=1, a,c\in\mathbb{Z}\right\}.$$

Since the equation of the plane tangent to the cone $z_1z_2-z^2=0$ through the point (z_1^0, z_2^0, z^0) is just $z_1z_2^0+z_2z_1^0-2zz^0=0$, the definition of $R_W(q)$ has a clear geometrical motivation.

Definition. Let $R(q)=R_W(q)\cup R_I^0(q)$ for any $q\in\bar{\mathbb{Q}}$. If $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{PGL}_2(\mathbb{Z})$ and $q=A/C\in\bar{\mathbb{Q}}$ then define the action $g\cdot q=\dfrac{aA+bC}{cA+dC}$.

It is easy to see that

$$R_W(g\cdot q)=g\cdot R_W(q)=gR_W(q)g^t. \tag{2.38}$$

Proposition 2.4. *For any $q\in\bar{\mathbb{Q}}$ the set $R(q)$ corresponds by v to a set of points in $\mathfrak{h}^*$ which form an affine root system of type $A_1^{(1)}$.*

Proof. See Sect. 3 and [23, 25] for details about the affine algebra of type $A_1^{(1)}$ and its root system. For our present purposes it suffices to know that such a root system is of the form

$$\{mv_0\pm v_1\,|\,m\in\mathbb{Z}\}\cup\{mv_0\,|\,0\neq m\in\mathbb{Z}\}, \tag{2.39}$$

where v_0 and v_1 are fixed vectors in $\mathfrak{h}^*$. From (2.36) and Proposition 2.1, and since for $q=\infty$ we have $a=1$ and $c=0$, we see that

$$R(\infty)=\left\{\begin{pmatrix}m & \pm 1\\ \pm 1 & 0\end{pmatrix}\middle|\, m\in\mathbb{Z}\right\}\cup\left\{\begin{pmatrix}m & 0\\ 0 & 0\end{pmatrix}\middle|\, 0\neq m\in\mathbb{Z}\right\}. \tag{2.40}$$

We will see later that this corresponds to the root system of the subalgebra $\mathfrak{F}_0$ of type $A_1^{(1)}$ generated by α_1 and α_2. The general result follows from the action of $\mathrm{PGL}_2(\mathbb{Z})$ on $\bar{\mathbb{Q}}$.

Proposition 2.5. *Each element $N\in v(R_W)$ is contained in exactly two sets $R_W(q_1)$ and $R_W(q_2)$ for $q_1,q_2\in\bar{\mathbb{Q}}$.*

Proof. This follows from Proposition 2.2.

The most subtle aspect of root systems of Kac-Moody algebras is the determination of the root multiplicities. The construction of algebra $\mathfrak{F}$ given in Sect. 4 provides a significant amount of such information.

In contrast to the cases of finite dimensional and affine algebras where the union of the weights of standard representations fill up the entire weight lattice P, for hyperbolic algebras they only fill up some positive cone which one denotes P^+.

Definition. Let $P^+\subset P$ be defined by

$$v(P^+)=\{N\in S_2'(\mathbb{Z})\,|\,N\geqq 0\}. \tag{2.41}$$

Within the set P^+ we will now study the orbits under the Weyl group W. This problem was studied by Lagrange in the language of binary quadratic forms.

Lagrange considered classes of integral binary quadratic forms

$$F(x,y)=ax^2+bxy+cy^2\quad\text{for}\quad a,b,c\in\mathbb{Z} \tag{2.42}$$

under the action of $\mathrm{PSL}_2(\mathbb{Z})$. There are finitely many classes of forms with given discriminant D. Each $N\in v(P^+)$ corresponds to a positive semi-definite form with discriminant $D\leqq 0$. If $\det(N)=0=D$ it is easy to find the $\mathrm{PSL}_2(\mathbb{Z})$-equivalent form $F_1(x,y)=nx^2$ for unique $n\in\mathbb{Z}^+$, so N is $\mathrm{PSL}_2(\mathbb{Z})$-conjugate to a unique element of $v(P^{++})$. To study the Weyl group orbits in P^+ it is clearly sufficient to study those

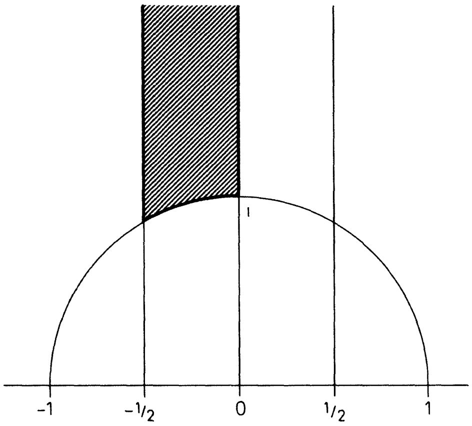


Fig. 1. Fundamental domain for $\mathrm{PGL}_2(\mathbb{Z})$

$0<N\in\nu(P^+)$ which are primitive. One has the well-known bijection between the set of all similarity classes of full modules in imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ having fixed coefficient ring $\mathfrak{O}_f$ and the set of all equivalence classes of positive definite integral primitive binary quadratic forms of discriminant $D=df^2<0$. (See [3] for details.) A representative module associated with $N=\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ can be taken with basis

$$\left\{1, \gamma=\frac{-b+\sqrt{b^2-4ac}}{2a}\right\}, \tag{2.43}$$

where $\mathrm{Im}(\gamma)>0$. Using the action of $g=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\in\mathrm{PGL}_2(\mathbb{Z})$ on the upper half-plane

$$g\cdot z=\begin{cases} \dfrac{\alpha z+\beta}{\gamma z+\delta} & \text{if} \quad \det(g)=+1 \\ \dfrac{\alpha\bar{z}+\beta}{\gamma\bar{z}+\delta} & \text{if} \quad \det(g)=-1 \end{cases} \tag{2.44}$$

one gets as fundamental domain

$$\{\gamma\in\mathbb{C}\,|\,\mathrm{Im}(\gamma)>0,\ -1/2\leqq\mathrm{Re}(\gamma)\leqq 0, |\gamma|\geqq 1\} \tag{2.45}$$

(see Fig. 1). These conditions applied to (2.43) give the conditions $c\geqq a\geqq b\geqq 0$ on N. Applying reflection r_3 we see that any $N\in\nu(P^+)$ is $\mathrm{PGL}_2(\mathbb{Z})$-conjugate to a unique element of $\nu(P^{++})$.

Proposition 2.6. *We have $P^+=W\cdot P^{++}$ and every W-orbit in P^+ contains exactly one element from P^{++}.*

Remark. The map T from $\nu(P^+)$ to the extended complex upper half-plane given by

$$T\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}=\frac{-b+\sqrt{b^2-4ac}}{2a} \tag{2.46}$$

is such that

$$T(xNx^t) = x \cdot T(N) \quad \text{for} \quad x \in \mathrm{PGL}_2(\mathbb{Z}). \tag{2.47}$$

This gives a computational proof of Proposition 2.6, and has been noticed by Yoshida [42] and used to investigate compactifications of certain quotient spaces.

3. Standard Representations and Characters of $\mathfrak{F}$ and Affine Subalgebra $\mathfrak{F}_0$

Let $\mathscr{E}$ be the space of all complex valued functions on P^+. Then $\mathscr{E}$ is a commutative algebra whose product is the convolution operation

$$(f_1 f_2)(\lambda) = \sum_{\mu \in P^+} f_1(\lambda - \mu) f_2(\mu). \tag{3.1}$$

Only finitely many terms in this sum are nonzero because P^+ is contained in half of a cone.

Let $e^\lambda \in \mathscr{E}$ be the function defined for any $\lambda \in P^+$ by

$$e^\lambda(\mu) = \begin{cases} 1 & \text{if} \quad \mu = \lambda \\ 0 & \text{if} \quad \mu \neq \lambda . \end{cases} \tag{3.2}$$

Then $e^\lambda e^\mu = e^{\lambda + \mu}$ and $e^0 = 1$ is the unit element in $\mathscr{E}$.

Let $\mathscr{C}$ be the category of $\mathfrak{F}$-modules V such that

$$V = \oplus \sum_{\mu \in P^+} V_\mu \quad \text{for} \quad \dim V_\mu < \infty . \tag{3.3}$$

Here we are denoting by

$$V_\mu = \{ v \in V \mid h \cdot v = \mu(h) v, \text{ for all } h \in \mathfrak{h} \} \tag{3.4}$$

the μ-weight space of V. It is easy to see that $\mathfrak{F}^\alpha \cdot V_\mu \subseteqq V_{\mu+\alpha}$ [recall the definition (2.4) of $\mathfrak{F}^\alpha$]. It is known that the action of the generators e_i and f_i [see (2.2)], $i = 1, 2, 3$ on V is locally nilpotent. Kac has shown in [16] that any $\mathfrak{F}$-module V in $\mathscr{C}$ is completely reducible. Any irreducible module in $\mathscr{C}$ has a dominant integral highest weight $\lambda \in P^{++}$. There is, in fact, a one-to-one correspondence between P^{++} and the collection of irreducible modules in $\mathscr{C}$ [15]. Thus, we label these so called standard modules, V^λ.

The character of any $\mathfrak{F}$-module V in $\mathscr{C}$ is defined to be

$$X(V) = \sum_{\mu \in P^+} \dim(V_\mu) e^\mu . \tag{3.5}$$

We usually denote $X(V^\lambda)$ by X^λ. Clearly $X(V) \in \mathscr{E}$.

Kac has established in [15] the character formula for V^λ analogous to the Weyl character formula for finite dimensional modules of finite dimensional semisimple Lie algebras.

Theorem 3.1. (Weyl-Kac character formula). *For $\lambda \in P^{++}$ we have*

$$X^\lambda = \frac{\sum_{w \in W} \det(w) e^{w(\lambda + \varrho)}}{\sum_{w \in W} \det(w) e^{w\varrho}}, \tag{3.6}$$

where $\varrho=\omega_1+\omega_2+\omega_3\in\mathfrak{h}^*$ *and where*

$$\det(w)=\begin{cases}+1 & \text{if} \quad w\in W^+\\ -1 & \text{if} \quad w\notin W^+.\end{cases} \tag{3.7}$$

Let $\mathfrak{F}_0$ denote the type $A_1^{(1)}$ Kac-Moody subalgebra of $\mathfrak{F}$ with generators e_i, f_i, h_i, $i=1,2$ and relations (2.2) whose Cartan matrix is therefore

$$\begin{pmatrix} 2 & -2\\ -2 & 2\end{pmatrix}. \tag{3.8}$$

It is known that this algebra is isomorphic to a one dimensional central extension, $\mathfrak{g}_0=\widetilde{\mathfrak{sl}_2(\mathbb{C})}\oplus\mathbb{C}c$, of

$$\widetilde{\mathfrak{sl}_2(\mathbb{C})}=\mathfrak{sl}_2(\mathbb{C})\otimes\mathbb{C}[t,t^{-1}]. \tag{3.9}$$

In order to overcome a problem caused by the degeneracy of the Cartan matrix (3.8) one can consider a bigger algebra, the semidirect product of $\mathfrak{g}_0$ with the one dimensional space spanned by the derivation $d=-t\frac{d}{dt}$. If we let

$$\mathfrak{g}_0^e=\widetilde{\mathfrak{sl}_2(\mathbb{C})}\oplus\mathbb{C}c\oplus\mathbb{C}d \tag{3.10}$$

be that extended algebra then the bracket products are as follows:

$$\begin{aligned}&[x\otimes t^n, y\otimes t^m]=[x,y]\otimes t^{n+m}+n\langle x,y\rangle\delta_{n,-m}c\\ \text{and}\quad &[d,x\otimes t^n]=-n(x\otimes t^n),\end{aligned} \tag{3.11}$$

where $x,y\in\mathfrak{sl}_2(\mathbb{C})$, $n,m\in\mathbb{Z}$, $\langle x,y\rangle=\frac{1}{4}\mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_y)$ is the Killing form on $\mathfrak{sl}_2(\mathbb{C})$ normalized so that $\langle h,h\rangle=2$, and c acts centrally.

We will identify $\mathfrak{g}_0^e$ as a subalgebra of $\mathfrak{F}$, $\mathfrak{F}_0^e\supset\mathfrak{F}_0$ as follows:

$$h\otimes 1=h_1, \quad c=h_1+h_2, \quad d=h_1+h_2+h_3, \tag{3.12}$$

$$e\otimes 1=e_1, \quad f\otimes 1=f_1, \quad f\otimes t=e_2, \quad e\otimes t^{-1}=f_2, \tag{3.13}$$

where $\{e,f,h\}$ is the usual basis of $\mathfrak{sl}_2(\mathbb{C})$. It is straightforward to verify the required relations. Note that (3.12) is the basis of $\mathfrak{h}_0^e$ identified with the basis of $\mathfrak{h}^*$

$$\gamma_1=\alpha_1, \quad \gamma_2=\alpha_1+\alpha_2, \quad \gamma_3=\alpha_1+\alpha_2+\alpha_3 \tag{3.14}$$

by the isomorphism η in (2.12). So the extended Cartan subalgebra $\mathfrak{h}_0^e\subset\mathfrak{g}_0^e$ has dual space $(\mathfrak{h}_0^e)^*\approx\mathfrak{h}^*$.

Let us use the following notation for elements of $\mathfrak{g}_0^e$:

$$e\otimes t^k=e(k), \quad f\otimes t^k=f(k), \quad h\otimes t^k=h(k), \tag{3.15}$$

for all $k\in\mathbb{Z}$. The Killing form $\langle\cdot,\cdot\rangle$ on $\mathfrak{sl}_2(\mathbb{C})$ extends uniquely to an invariant nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_0^e$ such that $\langle h_i,h_i\rangle=2$ for $1\leqq i\leqq 3$. One may easily compute that

$$\langle e(k),f(-k)\rangle=1, \quad \langle h(k),h(-k)\rangle=2, \quad \langle c,d\rangle=-1 \tag{3.16}$$

and all other values of the form on pairs of vectors from basis

$$\{e(k),f(k),h(k),c,d\,|\,k\in\mathbb{Z}\}$$

are zero. This follows from invariance and the fact that $\langle x, y\rangle = 0$ whenever x is an α root vector, y is a β root vector and $\alpha+\beta \neq 0$. We also know that $e(k)$ is a $(k+1)\alpha_1 + k\alpha_2$ root vector, $f(k)$ is a $(k-1)\alpha_1 + k\alpha_2$ root vector and $h(k)$ is a $k\alpha_1 + k\alpha_2$ root vector for all $k \in \mathbb{Z}$.

From (3.16) it is easy to see that the union of the following three sets of vectors is an orthonormal basis for $\mathfrak{g}_0^e$ with respect to form $\langle \cdot, \cdot \rangle$;

$$\left\{\frac{1}{\sqrt{2}}(e(k)+f(-k)), \frac{1}{\sqrt{-2}}(e(k)-f(-k)) \mid k \in \mathbb{Z}\right\}, \tag{3.17}$$

$$\left\{\frac{1}{2}(h(k)+h(-k)), \frac{1}{2\sqrt{-1}}(h(k)-h(-k)) \mid 0 < k \in \mathbb{Z}\right\}, \tag{3.18}$$

$$\left\{\frac{1}{\sqrt{2}}h(0), \frac{1}{\sqrt{-2}}(c+d), \frac{1}{\sqrt{2}}(c-d)\right\}. \tag{3.19}$$

Also, (3.19) is an orthonormal basis for $\mathfrak{h}_0^e$.

The results of Kac mentioned previously, e.g. classification of standard modules and the character formula, are quite general and apply as well to $\mathfrak{F}_0^e$. The theory of representations for the algebra $\mathfrak{F}_0$ requires only the ad-hoc adjunction of derivation d, but there appears to be some value in the consideration of $\mathfrak{F}_0$ in the context of $\mathfrak{F}$. However, the condition that an element $\lambda = n_1\gamma_1^* + n_2\gamma_2^* + t\gamma_3^* \in \mathfrak{h}^*$ be integral (respectively, dominant integral) for the algebra $\mathfrak{F}_0^e$ means only that $n_1, n_2 \in \mathbb{Z}$ (respectively, $n_1, n_2 \in \mathbb{Z}^+$) while t may vary over $\mathbb{C}$.

Fundamental weights of $\mathfrak{F}_0^e$ are defined by $\lambda_i(h_j) = \delta_{ij}$ for $i = 1, 2$, which imposes no condition on the γ_3^* coefficient of λ_i, so there is a one parameter family of such weights. Define

$$\omega_1^t = \gamma_1^* + \gamma_2^* + t\gamma_3^*, \tag{3.20}$$

$$\omega_2^t = \gamma_2^* + t\gamma_3^*, \tag{3.21}$$

and

$$\omega_3^t = t\gamma_3^*. \tag{3.22}$$

The weights ω_3^t are the dominant weights of $\mathfrak{F}_0^e$ whose standard modules are the one dimensional trivial modules. Note that

$$\omega_1^1 = \omega_1, \quad \omega_2^1 = \omega_2 \quad \text{and} \quad \omega_3^1 = \omega_3. \tag{3.23}$$

The root lattice of $\mathfrak{F}_0^e$ is

$$Q_0 = \{2n_1\gamma_1^* + n_3\gamma_3^* \mid n_1, n_3 \in \mathbb{Z}\} \tag{3.24}$$

and the weight "lattice" of $\mathfrak{F}_0^e$ is

$$P_0 = \{n_1\gamma_1^* + n_2\gamma_2^* + t\gamma_3^* \mid n_1, n_2 \in \mathbb{Z}, t \in \mathbb{C}\}. \tag{3.25}$$

The Weyl group W_0 of $\mathfrak{F}_0^e$ is an infinite dihedral group with generators r_1, r_2 whose action on $\mathfrak{h}^*$ is defined by

$$r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i \quad \text{for} \quad i = 1, 2. \tag{3.26}$$

So we have

$$r_1(n_1\gamma_1^* + n_2\gamma_2^* + t\gamma_3^*) = -n_1\gamma_1^* + n_2\gamma_2^* + t\gamma_3^*, \tag{3.27}$$

and

$$r_2(n_1\gamma_1^*+n_2\gamma_2^*+t\gamma_3^*)=(2n_2-n_1)\gamma_1^*+n_2\gamma_2^*+(n_2+t-n_1)\gamma_3^*. \tag{3.28}$$

Note that W_0 fixes γ_3^* so for any $w_0\in W_0$, $\lambda\in\mathfrak{h}^*$, $w_0(\lambda+t\gamma_3^*)=w_0(\lambda)+t\gamma_3^*$. It therefore suffices to compute the action of W_0 only on the span of γ_1^* and γ_2^*.

The translations in W_0 are the powers of $T=r_1r_2$. One has

$$T^i(n_1\gamma_1^*+n_2\gamma_2^*)=(2in_2+n_1)\gamma_1^*+n_2\gamma_2^*+(i^2n_2+in_1)\gamma_3^*. \tag{3.29}$$

Define $\mathscr{C}_0$ to be the category of $\mathfrak{F}_0^e$-modules which decompose into a direct sum of $\mathfrak{F}_0^e$ weight spaces each of which is finite dimensional. Define $\mathscr{E}_0$ to be the algebra of complex valued functions on

$$P_0^+=\{\lambda=n_1\gamma_1^*+n_2\gamma_2^*+t\gamma_3^*\in P_0\,|\,(\lambda,\lambda)\leqq 0, n_2\geqq 0, t\geqq 0\} \tag{3.30}$$

with convolution product. Then $\mathscr{E}\subseteqq\mathscr{E}_0$, the difference being that the γ_3^* coefficient is allowed to vary continuously over $\mathbb{R}^+$ for elements of $\mathscr{E}_0$, but must be in $\mathbb{Z}^+$ for those in $\mathscr{E}$.

Kac [15] generalized the classical Casimir element, which can be written [8] as

$$\sum_{i=1}^{\ell} h_i^2+2\sum_{\alpha>0} x_{-\alpha}x_\alpha+2h_\varrho. \tag{3.31}$$

From the condition that $\langle h_\varrho, h_i\rangle=1$ for $1\leqq i\leqq 3$ we find that for $\mathfrak{F}_0^e$

$$h_\varrho=-\tfrac{9}{2}h_1-5h_2-2h_3=\tfrac{1}{2}h(0)-3c-2d. \tag{3.32}$$

From (3.16), (3.19), (3.31), and (3.32) we see that the Casimir element of $\mathfrak{F}_0^e$ can be written as

$$C=\tfrac{1}{2}h(0)^2-2cd+2\sum_{k\geqq 0} f(-k)e(k)+2\sum_{k>0} e(-k)f(k)$$
$$+\sum_{k>0} h(-k)h(k)+h(0)-6c-4d. \tag{3.33}$$

If V^λ is a standard irreducible $\mathfrak{F}_0^e$-module with highest weight $\lambda=n_1\gamma_1^*+n_2\gamma_2^*+n_3\gamma_3^*$ then the Casimir element C acts on V^λ as a scalar which we will now compute. From (3.12) we see that basis $\{\gamma_1^*,\gamma_2^*,\gamma_3^*\}$ of $\mathfrak{h}^*=(\mathfrak{h}_0^e)^*$ is dual to basis $\{h(0),c,d\}$ of $\mathfrak{h}=\mathfrak{h}_0^e$, so $\lambda(h(0))=n_1$, $\lambda(c)=n_2$ and $\lambda(d)=n_3$. Since the action of C commutes with the action of $\mathfrak{F}_0^e$ on V^λ, it suffices to compute $C\cdot v_\lambda^+$, where v_λ^+ is a highest weight vector of V^λ. Any element of $\mathfrak{F}_0^e$ from a positive root space kills v_λ^+, so

$$\begin{aligned} C\cdot v_\lambda^+ &=(\tfrac{1}{2}h(0)^2+h(0)-6c-4d-2cd)\cdot v_\lambda^+\\ &=(\tfrac{1}{2}n_1^2+n_1-6n_2-4n_3-2n_2n_3)v_\lambda^+\\ &=\langle\lambda+2\varrho,\lambda\rangle v_\lambda^+, \end{aligned} \tag{3.34}$$

where

$$\varrho=\omega_1+\omega_2+\omega_3=\gamma_1^*+2\gamma_2^*+3\gamma_3^*. \tag{3.35}$$

The standard $\mathfrak{F}_0^e$-modules have been stratified according to their "level" by several authors [6, 16, 23]. The level of V^λ for the λ given above is n_2. For standard highest weight modules this is always a nonnegative integer. The only standard modules on level 0 are trivial one dimensional modules. Standard $\mathfrak{F}_0^e$-modules whose highest weights differ only by a multiple of $\gamma_3^* = \omega_3$ are isomorphic as $\mathfrak{F}_0$-modules, that is, they are only distinguished by the action of derivation d. Their characters differ only by a factor of $e^{t\omega_3}$ for some t.

For the affine Kac-Moody algebras a certain standard module on level 1 called the basic module plays a special role in the theory [7–9] and admits a simple construction in many cases [25, 10, 17]. For $\mathfrak{F}_0^e$ the basic module has highest weight $-\alpha_3 = \omega_2^0 - \omega_3$, whose ω_3 coefficient has been chosen because of its relevance to the structure of $\mathfrak{F}$. We will shortly discuss the construction of this basic module whose character was first computed in [6], and whose weight multiplicities were found to be the values of the classical partition function. We shall denote the character of a standard $\mathfrak{F}_0^e$-module V^λ by χ^λ. From [6] we have the following character formulas: (cf. [16])

$$\chi^{\omega_1} = e^{\omega_1}\Big(\sum_{n \geqq 0} p(n) e^{n\omega_3}\Big)\Big(\sum_{m \in \mathbb{Z}} e^{m\alpha_1 + m(m+1)\omega_3}\Big) \tag{3.36}$$

and

$$\chi^{\omega_2} = e^{\omega_2}\Big(\sum_{n \geqq 0} p(n) e^{n\omega_3}\Big)\Big(\sum_{m \in \mathbb{Z}} e^{m\alpha_1 + m^2\omega_3}\Big), \tag{3.37}$$

where p is the classical partition function of elementary number theory.

We may rewrite these as

$$\chi^{\omega_1} = e^{\gamma_2^* + \frac{3}{4}\gamma_3^*}\Big(\sum_{n \geqq 0} p(n) e^{n\gamma_3^*}\Big)\Big(\sum_{m \in \mathbb{Z}} e^{2\left(m+\frac{1}{2}\right)\gamma_1^* + \left(m+\frac{1}{2}\right)^2\gamma_3^*}\Big) \tag{3.38}$$

and

$$\chi^{\omega_2} = e^{\gamma_2^* \gamma_3^*}\Big(\sum_{n \geqq 0} p(n) e^{n\gamma_3^*}\Big)\Big(\sum_{m \in \mathbb{Z}} e^{2m\gamma_1^* + m^2\gamma_3^*}\Big). \tag{3.39}$$

Using the Weyl-Kac character formula and the technique of "principal specialization" one recovers from these formulas [6] the result of Gauss

$$\frac{\phi(q^2)^2}{\phi(q)} = \sum_{j \in \mathbb{Z}} q^{j(2j+1)}, \tag{3.40}$$

where

$$\phi(q) = \prod_{n \geqq 1} (1 - q^n). \tag{3.41}$$

There are two approaches to the study of standard modules on levels greater than 1. One approach is to restrict the basic representation to the subalgebra $\mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t^n, t^{-n}] \otimes \mathbb{C}c$ which is isomorphic to $\mathfrak{g}_0$. The basic module is then reducible and contains standard $\mathfrak{g}_0$ modules of level n [9]. Another approach to the higher levels is by studying tensor products of modules from level 1 [5, 16]. We will later require the following decomposition of the basic $\mathfrak{F}_0^e$-module tensored with itself, which can be found in [5] or [16]:

$$V^{\omega_2^0 - \omega_3} \otimes V^{\omega_2^0 - \omega_3} = \sum_{m \geqq 0} (a_m e^{(m-2)\omega_3} V^{2\omega_2^0} + b_m e^{(m-1)\omega_3} V^{2\omega_1^0}), \tag{3.42}$$

where

$$\sum_{m\geq 0}(a_m x^{2m}+b_m x^{2m+1})=\prod_{j\geq 1}(1+x^{2j-1}). \tag{3.43}$$

The elementary techniques used in [6] are able to give the characters of level 2 and 3 standard $\mathfrak{F}_0^e$-modules. We will later need the following formulas:

$$\chi^{2\omega_1^0}=e^{2\gamma_2^*}t^{-1}\left[\left(\sum_{m\geq 0}E(2m)t^{2m}\right)\left(\sum_{k\in\mathbb{Z}}e^{(2k+1)\alpha_1}t^{(2k+1)^2}\right)\right.$$
$$\left.+\left(\sum_{m\geq 0}E(2m+1)t^{2m+1}\right)\left(\sum_{k\in\mathbb{Z}}e^{2k\alpha_1}t^{4k^2}\right)\right], \tag{3.44}$$

where $t^2=q$ and the weight multiplicities are given by

$$\sum_{m\geq 0}E(m)t^m=\prod_{j\geq 1}(1-t^{4j})^{-1}(1-t^{4j-1})^{-1}(1-t^{4j-3})^{-1}, \tag{3.45}$$

and

$$\chi^{2\omega_2^0}=e^{2\gamma_2^*}\left[\left(\sum_{m\geq 0}E(2m)t^{2m}\right)\left(\sum_{k\in\mathbb{Z}}e^{2k\alpha_1}t^{4k^2}\right)\right.$$
$$\left.+\left(\sum_{m\geq 0}E(2m+1)t^{2m+1}\right)\left(\sum_{k\in\mathbb{Z}}e^{(2k+1)\alpha_1}t^{(2k+1)^2}\right)\right]. \tag{3.46}$$

For brevity let us denote the basic module $V^{\omega_2^0-\omega_3}$ by V. The tensor product (3.42) has an obvious decomposition into two "strings" of $\mathfrak{F}_0$-isomorphic modules

$$\sum_{m\geq 0}(a_m q^{(m-2)})V^{2\omega_2^0}\quad\text{and}\quad\sum_{m\geq 0}(b_m q^{(m-1)})V^{2\omega_1^0}, \tag{3.47}$$

where $q=e^{\omega_3}$. There is also a natural decomposition into the symmetric tensors

$$S(V)=\{v_1\otimes v_2+v_2\otimes v_1\mid v_1,v_2\in V\} \tag{3.48}$$

and the antisymmetric tensors

$$A(V)=\{v_1\otimes v_2-v_2\otimes v_1\mid v_1,v_2\in V\}, \tag{3.49}$$

both of which are reducible $\mathfrak{g}_0^e$-modules. We wish to establish

Theorem 3.2.

$$S(V)=\sum_{m\geq 0}a_m e^{(m-2)\omega_3}V^{2\omega_2^0} \tag{3.50}$$

and

$$A(V)=\sum_{m\geq 0}b_m e^{(m-1)\omega_3}V^{2\omega_1^0}. \tag{3.51}$$

We recall first the construction of the basic $\mathfrak{g}_0^e$-module V given by Lepowsky and Wilson [25]. Define the elements

$$h'(k)=e\otimes t^{(k-1)/2}+f\otimes t^{(k+1)/2}\quad\text{for}\quad k\in 2\mathbb{Z}+1, \tag{3.52}$$

and let

$$x(n)=\begin{cases}h\otimes t^{n/2}-\frac{1}{2}\delta_{0,n}c & \text{if}\quad n\in 2\mathbb{Z}\\ -e\otimes t^{(n-1)/2}+f\otimes t^{(n+1)/2} & \text{if}\quad n\in 2\mathbb{Z}+1.\end{cases} \tag{3.53}$$

Then from (3.11) one easily finds that

$$[h'(j), h'(k)] = j\delta_{j,-k}c \quad \text{for} \quad j, k \in 2\mathbb{Z}+1, \tag{3.54}$$
$$[h'(k), x(n)] = 2x(n+k) \quad \text{for} \quad k \in 2\mathbb{Z}+1 \quad \text{and} \quad n \in \mathbb{Z}, \tag{3.55}$$

and

$$[x(k), x(n)] = \begin{cases} k\delta_{k,-n}c & \text{if} \quad k, n \in 2\mathbb{Z} \\ -k\delta_{k,-n}c & \text{if} \quad k, n \in 2\mathbb{Z}+1 \\ 2h'(k+n) & \text{if} \quad k \in 2\mathbb{Z}+1, n \in 2\mathbb{Z}. \end{cases} \tag{3.56}$$

The brackets of derivation d with these elements are easy to compute. A basis for $\mathfrak{g}_0^e$ consists of

$$\{c, d, h'(k), x(n) \mid k \in 2\mathbb{Z}+1, n \in \mathbb{Z}\}. \tag{3.57}$$

The subalgebra

$$\mathfrak{h}' = \sum_{k \in 2\mathbb{Z}+1} \mathbb{C}h'(k) + \mathbb{C}c \tag{3.58}$$

is called the principal Heisenberg subalgebra of $\mathfrak{g}_0^e$, and the basic $\mathfrak{g}_0^e$-module V is an irreducible $\mathfrak{h}'$-module. One can identify V with the symmetric algebra

$$\mathfrak{S} = \mathfrak{S}(h'(-1), h'(-3), h'(-5), \ldots) \tag{3.59}$$

of polynomials in $\{h'(-j) \mid j \in 2\mathbb{Z}^+ + 1\}$, where the highest weight vector $v_0 \in V$ corresponds to $1 \in \mathfrak{S}$. The action of $h'(-k)$ for $k \in 2\mathbb{Z}^+ + 1$ is by multiplication on the left, that of $h'(k)$ for $k \in 2\mathbb{Z}^+ + 1$ is by the derivation $k\dfrac{\partial}{\partial h'(-k)}$, and c acts as the identity. The action of $x(n)$ for $n \in \mathbb{Z}$ is given by a vertex operator

$$X(2h', z) = -\frac{1}{2}\exp\left(\sum_{k \in 2\mathbb{Z}^+ + 1} \frac{z^k}{k} 2h'(-k)\right) \exp\left(-\sum_{k \in 2\mathbb{Z}^+ + 1} \frac{z^{-k}}{k} 2h'(k)\right) \tag{3.60}$$

in the following way. For $n \in \mathbb{Z}$ let $X_n(2h')$ denote the n^{th} homogeneous component of $X(2h', z)$, so that

$$X(2h', z) = \sum_{n \in \mathbb{Z}} X_n(2h') z^{-n}. \tag{3.61}$$

Then the operators $X_n(2h')$ are well-defined on $\mathfrak{S}$, satisfy the commutation relations (3.55) and (3.56), and act on $\mathfrak{S}$ as the operators $x(n)$ act on V.

Now consider the tensor product $V \otimes V$. We will use subscripts 1 and 2 to indicate the first and second factor, respectively. So we will let

$$h_1'(k) = h'(k) \otimes 1 \quad \text{and} \quad h_2'(k) = 1 \otimes h'(k) \quad \text{for} \quad k \in 2\mathbb{Z}+1 \tag{3.62}$$

whose action on $v_1 \otimes v_2$ is given by

$$h_1'(k) \cdot (v_1 \otimes v_2) = (h'(k) \cdot v_1) \otimes v_2 \tag{3.63}$$

and

$$h_2'(k) \cdot (v_1 \otimes v_2) = v_1 \otimes (h'(k) \cdot v_2). \tag{3.64}$$

Then the action of $\mathfrak{g}_0^e$ on $V \otimes V$ is given by the operators $h'_1(k)+h'_2(k)$ for $k \in 2\mathbb{Z}+1$, $X_n(2h'_1)+X_n(2h'_2)$ for $n \in \mathbb{Z}$, and c acts as the scalar 2.

We introduce the following auxiliary vertex operator on $V \otimes V$,

$$X(h'_1-h'_2, z) = \exp\left(\sum_{k \in 2\mathbb{Z}^+ + 1} \frac{z^k}{k}(h'_1(-k)-h'_2(-k))\right) \cdot \exp\left(-\sum_{k \in 2\mathbb{Z}^+ + 1} \frac{z^{-k}}{k}(h'_1(k)-h'_2(k))\right) \tag{3.65}$$

with n^{th} homogeneous component $X_n(h'_1-h'_2)$, so that

$$X(h'_1-h'_2, z) = \sum_{n \in \mathbb{Z}} X_n(h'_1-h'_2) z^{-n}. \tag{3.66}$$

Let us denote anti-commutators by $\{A, B\} = AB+BA$. Then direct calculations with vertex operators show that for $m \in \mathbb{Z}$, $n \in 2\mathbb{Z}+1$ we have

$$\{X_n(h'_1-h'_2), X_m(h'_1-h'_2)\} = -2\delta_{n,-m}, \tag{3.67}$$

$$\{X_n(h'_2-h'_1), X_m(2h'_1)+X_m(2h'_2)\} = 0, \tag{3.68}$$

and

$$X_m(h'_1-h'_2)\cdot(1 \otimes 1) = 0 \quad \text{for} \quad m > 0. \tag{3.69}$$

For example, in the proof of (3.68) one uses

$$\{X_n(h'_2-h'_1), X_m(2h'_1)\} = \frac{1}{2\pi i}\int_{C_0}\left(\frac{1}{2\pi i}\int_{C_R} X(h'_2-h'_1, z)X(2h'_1, w)z^{n-1}\,dz + \frac{1}{2\pi i}\int_{C_r} X(2h'_1, w)X(h'_2-h'_1, z)z^{n-1}\,dz\right)w^{m-1}\,dw, \tag{3.70}$$

where $r < r_0 < R$ and C_0 is a fixed circle of radius r_0. Elementary operations then give

$$\frac{1}{2\pi i}\int_{C_0} 2X(h'_2+h'_1, w)w^{n+m-1}\,dw = 2X_{n+m}(h'_2+h'_1), \tag{3.71}$$

and by symmetry we have

$$\{X_n(h'_1-h'_2), X_m(2h'_2)\} = 2X_{n+m}(h'_1+h'_2). \tag{3.72}$$

These give the result. Details concerning vertex operators can be found in [8, 10].

Let $\Omega \subset V \otimes V$ be the subspace of highest weight vectors for $\mathfrak{g}_0^e$. So Ω is killed by the space of positive root vectors from $\mathfrak{g}_0^e$ which is spanned by

$$\{x(n), h'(k) \mid 0 < n \in \mathbb{Z}, 0 < k \in 2\mathbb{Z}+1\}. \tag{3.73}$$

In $\mathfrak{S} \otimes \mathfrak{S}$ this corresponds to the space of tensors killed by

$$\{X_n(2h'_1)+X_n(2h'_2), h'_1(k)+h'_2(k) \mid 0 < n \in \mathbb{Z}, 0 < k \in 2\mathbb{Z}+1\}. \tag{3.74}$$

From (3.42) we have the character

$$\chi(\Omega) = \sum_{m \geqq 0} (a_m e^{(m-2)\omega_3 + 2\omega_2^0} + b_m e^{(m-1)\omega_3 + 2\omega_1^0}), \tag{3.75}$$

where the coefficients are given by (3.43). Let $\mathrm{ch}(\cdot)$ denote the principally specialized character of any $\mathfrak{g}_0^e$-module, that is, the result of setting $e^{-\alpha_1}=u=e^{-\alpha_2}$ in the character. Then we have

$$\mathrm{ch}(\Omega)=e^{-2\alpha_3}\prod_{j\geqq 1}(1+u^{2j-1}). \tag{3.76}$$

Consider the set of vectors

$$X_{-2n_1-1}(h_1'-h_2')X_{-2n_2-1}(h_1'-h_2')\dots X_{-2n_k-1}(h_1'-h_2')\cdot(1\otimes 1), \tag{3.77}$$

where $n_1>n_2>\dots>n_k\geqq 0$. Using (3.67), (3.68), and (3.69) it is easy to prove that these vectors are linearly independent and are killed by $X_n(2h_1')+X_n(2h_2')$ for $0<n\in\mathbb{Z}$. If we show that they are also killed by $h_1'(k)+h_2'(k)$ for $0<k\in 2\mathbb{Z}+1$ then (3.76) shows that they are a basis for Ω. It is clear from (3.65) that $h_1'(k)+h_2'(k)$ and $X_n(h_1'-h_2')$ commute. Thus, $h_1'(k)+h_2'(k)$ applied to (3.77) will give 0 when $0<k\in 2\mathbb{Z}+1$.

Note that the vector in (3.77) is symmetric if it is unchanged when $h_1'-h_2'$ is replaced by $h_2'-h_1'$, and antisymmetric if the sign reverses. Because the subscripts are odd we see that the vector is symmetric when k is even and antisymmetric when k is odd.

Consider the element $x(0)=h\otimes 1-\frac{1}{2}c$ of the Cartan subalgebra of $\mathfrak{g}_0^e$, whose action on $V\otimes V$ is given by the operator $X_0(2h_1')+X_0(2h_2')$. If $v\in V$ has weight $\omega_1^0-m\omega_3$ for some m then

$$x(0)\cdot v=\tfrac{1}{2}v, \tag{3.78}$$

but if $v\in V$ has weight $\omega_2^0-m\omega_3$ then

$$x(0)\cdot v=-\tfrac{1}{2}v. \tag{3.79}$$

Thus, for any vector $v\in\Omega$ of the form (3.77), either v is of weight $2\omega_1^0-m\omega_3$ for some $m\in\mathbb{Z}$ and

$$(X_0(2h_1')+X_0(2h_2'))\cdot v=v, \tag{3.80}$$

or else v is of weight $2\omega_2^0-m\omega_3$ and

$$(X_0(2h_1')+X_0(2h_2'))\cdot v=-v. \tag{3.81}$$

But (3.68) with $m=0$ implies that in the first case k is odd and in the second case k is even because $1\otimes 1$ has weight $2\omega_2^0-\omega_3$. This completes the proof of Theorem 3.2.

Remark. The above result is closely related to Theorem 1.6 of [9] for the case when $n=m=2$.

Note that Theorem 3.2 along with (3.43)–(3.46) allow one to easily compute the characters of the symmetric and antisymmetric parts of the tensor product $V\otimes V$. Using (3.40) and (3.41) it is easy to obtain the principally specialized characters $\mathrm{ch}(S(V))$ and $\mathrm{ch}(A(V))$.

Corollary 3.3. *We have the principally specialized characters*

$$\mathrm{ch}(S(V))=e^{2\gamma_1^0}\frac{u^{-4}}{2}\frac{\phi(u^4)}{\phi(u)}\left[\prod_{j\geqq 1}(1+u^{2j-1})+\prod_{j\geqq 1}(1-u^{2j-1})\right] \tag{3.82}$$

and

$$\operatorname{ch}(A(V)) = e^{2\gamma_2^*} \frac{u^{-4}}{2} \frac{\phi(u^4)}{\phi(u)} \left[\prod_{j \geqq 1} (1+u^{2j-1}) - \prod_{j \geqq 1} (1-u^{2j-1}) \right]. \tag{3.83}$$

From (3.34) we have the scalar by which the Casimir element C acts on an irreducible standard module V^λ. Let us consider the two "strings" of modules in (3.42). If $\lambda = 2\gamma_2^* + n_3\gamma_3^* = 2\omega_2^0 + n_3\omega_3$ then (3.34) says

$$C \cdot v_\lambda = (-12 - 8n_3) v_\lambda \tag{3.84}$$

for any $v_\lambda \in V^\lambda$, while if $\lambda = 2\gamma_1^* + 2\gamma_2^* + n_3\gamma_3^* = 2\omega_1^0 + n_3\omega_3$ then

$$C \cdot v_\lambda = -8(n_3+1) v_\lambda . \tag{3.85}$$

In the first case, since $n_3 \in \mathbb{Z}$, the scalar is never zero, but in the second case it is zero if and only if $n_3 = -1$, that is, when $\lambda = -2\alpha_3 - \alpha_2$. The coefficient in (3.42) which corresponds to this λ is $b_0 = 1$, so there is a unique irreducible component of $V \otimes V$ on which C acts as zero.

Corollary 3.4. *In the tensor product $V \otimes V$ of the basic module of $\mathfrak{F}_0^e$ with itself, the subspace of vectors killed by the Casimir element C forms an irreducible standard module with highest weight $2\omega_1^0 - \omega_3$.*

For reasons which will not be apparent until later we would like to compute the principally specialized character of the antisymmetric "string" $A(V)$ with the first standard module (the one referred to in Corollary 3.4) removed.

Corollary 3.5. *The principally specialized character of*

$$\left(\sum_{m \geqq 1} b_m q^{m-1} \right) V^{2\omega_1^0}, \tag{3.86}$$

where the coefficients b_m are given by (3.43), *is*

$$e^{2\gamma_2^*} \frac{u^{-4}}{2} \frac{\phi(u^4)}{\phi(u)} \left[\prod_{j \geqq 1} (1+u^{2j-1}) - \prod_{j \geqq 1} (1-u^{2j-1}) - 2u \right]. \tag{3.87}$$

We now describe how characters in $\mathscr{E}$ can be viewed as functions of three complex variables and their domain of definition.

With respect to the bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ the basis of $\mathfrak{h}^*$ dual to basis (2.13) is

$$\gamma_1 = \alpha_1 , \quad \gamma_2 = \alpha_1 + \alpha_2 , \quad \gamma_3 = \alpha_1 + \alpha_2 + \alpha_3 . \tag{3.88}$$

If we write

$$z = z_1\gamma_1 + z_2\gamma_2 + z_3\gamma_3 \tag{3.89}$$

then for

$$\lambda = n_1\gamma_1^* + n_2\gamma_2^* + n_3\gamma_3^* \in P \tag{3.90}$$

we have

$$(\lambda, z) = n_1 z_1 + n_2 z_2 + n_3 z_3 . \tag{3.91}$$

For each $\lambda \in P^+$ we make $e^\lambda \in \mathscr{E}$ a function of z_1, z_2, z_3 by defining

$$e^\lambda(z_1, z_2, z_3) = e^{2\pi i(\lambda, z)} = e^{2\pi i(n_1 z_1 + n_2 z_2 + n_3 z_3)}. \tag{3.92}$$

For any character $X = \sum_{\lambda \in P^+} a(\lambda) e^\lambda \in \mathscr{E}$ we define

$$X(z_1, z_2, z_3) = \sum_{\lambda \in P^+} a(\lambda) e^\lambda(z_1, z_2, z_3) \tag{3.93}$$

for $z_j = x_j + iy_j \in \mathbb{C}$ satisfying the conditions $y_2 y_3 > y_1^2$, $y_2, y_3 > 0$. The function $X(z) = X(z_1, z_2, z_3)$ is defined on

$$\mathfrak{D} = \{z = z_1\gamma_1 + z_2\gamma_2 + z_3\gamma_3 \in \mathfrak{h}^* | z_j = x_j + iy_j, y_2 y_3 > y_1^2, y_2 > 0\}. \tag{3.94}$$

Then

$$v(\mathfrak{D}) = \left\{\mathfrak{Z} = \begin{pmatrix} -z_2 & z_1 \\ z_1 & -z_3 \end{pmatrix} \in S_2(\mathbb{C}) | \operatorname{Im}(z_j) = y_j, y_2 y_3 > y_1^2, y_2 > 0\right\}. \tag{3.95}$$

From our point of view this is natural because v induces a form on $S_2(\mathbb{C})$ such that

$$(v(\lambda), v(z)) = n_1 z_1 + n_2 z_2 + n_3 z_3. \tag{3.96}$$

It is more traditional, however, in the theory of Siegel modular forms (see [1]) to consider the set $v(P^+)$ as we have done, but to use instead of $v(\mathfrak{D})$ the set

$$v(\mathfrak{D})^\iota = \left\{\mathfrak{Z} = \begin{pmatrix} z_3 & z_1 \\ z_1 & z_2 \end{pmatrix} \in S_2(\mathbb{C}) | \operatorname{Im}(\mathfrak{Z}) > 0\right\}, \tag{3.97}$$

where ι denotes the "cannonical involution"

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\iota = \begin{pmatrix} -d & c \\ b & -a \end{pmatrix} \tag{3.98}$$

which satisfies $x^\iota y^\iota = -(xy)^\iota$ and which may be realized in $\mathfrak{h}^*$ as $-Ir_1 r_3$. The set given in (3.97) is called the Siegel domain (or Siegel upper half-plane) H_2 of genus 2. It is standard in the theory of Siegel modular forms to consider functions on H_2 which can be written as

$$\sum_{N \geqq 0} A(N) e^{2\pi i \operatorname{Tr}(N\mathfrak{Z})} \tag{3.99}$$

for $N \in v(P^+)$, $\mathfrak{Z} \in v(\mathfrak{D})^\iota = H_2$ and where

$$\operatorname{Tr}(N\mathfrak{Z}) = n_1 z_1 + n_2 z_2 + n_3 z_3. \tag{3.100}$$

This traditional use of the trace appears to obscure the indefinite bilinear form on $S_2(\mathbb{C})$ which from our point of view naturally gives rise to the form $n_1 z_1 + n_2 z_2 + n_3 z_3$ and gives a connection between the theory of Kac-Moody algebras and that of Siegel modular forms.

In order to work in the traditional domain H_2 only minor adjustments are needed. Because

$$(gxg^t)^\iota = g^\iota x^\iota (g^\iota)^t, \tag{3.101}$$

for $g \in \mathrm{PGL}_2(\mathbb{Z})$ the action of $\mathrm{PGL}_2(\mathbb{Z})$ on $x^\iota \in H_2$ is given by

$$g \cdot x^\iota = g^\iota x^\iota (g^\iota)^t. \tag{3.102}$$

4. The Construction of $\mathfrak{F}$

We shall now show how the algebra $\mathfrak{F}$ can be constructed from the subalgebra $\mathfrak{F}_0^e$, its basic module $V=V^{-\alpha_3}$ and the dual contragredient module V^*. This construction, inspired by the work of Kantor [19], gives $\mathfrak{F}$ as a $\mathbb{Z}$-graded algebra

$$\ldots+\mathfrak{F}_{-2}+\mathfrak{F}_{-1}+\mathfrak{F}_0^e+\mathfrak{F}_1+\mathfrak{F}_2+\ldots, \tag{4.1}$$

where $\mathfrak{F}_{-1}\approx V$ and $\mathfrak{F}_1\approx V^*$, whose weight multiplicities were shown in [6] to be values of the classical partition function p.

Let $V=V^{\omega_2^0-\omega_3}=V^{-\alpha_3}$ be the basic $\mathfrak{g}_0^e$-module with highest weight vector v_0 and let V^* be the dual space of linear functionals on V. We will write the value of the functional $v^*\in V^*$ on $v\in V$ as $\langle v^*|v\rangle$. The contragredient action of $x\in\mathfrak{g}_0^e$ on v^* is determined by

$$\langle x\cdot v^*|v\rangle=-\langle v^*|x\cdot v\rangle \tag{4.2}$$

which implies that

$$[x,y]\cdot v^*=x\cdot(y\cdot v^*)-y\cdot(x\cdot v^*) \tag{4.3}$$

for $x,y\in\mathfrak{g}_0^e$, so V^* is a $\mathfrak{g}_0^e$-module. We will define $v^*\cdot x=-x\cdot v^*$ so that $\langle v^*\cdot x|v\rangle=\langle v^*|x\cdot v\rangle$ may be simply written $\langle v^*xv\rangle$. Then V^* is an irreducible lowest weight $\mathfrak{g}_0^e$-module with lowest weight vector v_0^* of weight α_3. One has

$$\begin{aligned}&e_1\cdot v_0=0, e_2\cdot v_0=0,\\&h_1\cdot v_0=0, h_2\cdot v_0=v_0, h_3\cdot v_0=-2v_0,\\&f_1\cdot v_0=0, f_2\cdot(f_2\cdot v_0)=0,\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}&f_1\cdot v_0^*=0, f_2\cdot v_0^*=0,\\&h_1\cdot v_0^*=0, h_2\cdot v_0^*=-v_0^*, h_3\cdot v_0^*=2v_0^*,\\&e_1\cdot v_0^*=0, e_2\cdot(e_2\cdot v_0^*)=0.\end{aligned} \tag{4.5}$$

Define a map $\phi: V^*\times V\to\mathfrak{g}_0^e$ by

$$\phi(v^*,v)=-\sum_{i\in I}\langle v^*x_iv\rangle x_i \tag{4.6}$$

where $\{x_i|i\in I\}$ is an orthonormal basis for $\mathfrak{g}_0^e$ with respect to the form $\langle\cdot,\cdot\rangle$. It is clear that only finitely many terms in this sum are nonzero for any particular pair (v^*,v), and that the definition of ϕ is independent of choice of orthonormal basis.

Proposition 4.1. *For any $x\in\mathfrak{g}_0^e$, $v\in V$, $v^*\in V^*$ we have*

$$[x,\phi(v^*,v)]=\phi(x\cdot v^*,v)+\phi(v^*,x\cdot v).$$

Proof. Let $[x_i,x_j]=\sum_{k\in I}C_{ij}^kx_k$ where $\{x_i|i\in I\}$ is an orthonormal basis for $\mathfrak{g}_0^e$. The invariance of the form, $\langle[x_i,x_j],x_k\rangle=\langle x_i,[x_j,x_k]\rangle$, says $C_{ij}^k=C_{jk}^i$. So we have

$$\begin{aligned}[x_j,\phi(v^*,v)]&=-\sum_{i\in I}\langle v^*x_iv\rangle[x_j,x_i]\\&=-\sum_{i\in I}\sum_{k\in I}\langle v^*x_iv\rangle C_{ji}^kx_k=-\sum_{i\in I}\sum_{k\in I}\langle v^*x_kv\rangle C_{jk}^ix_i.\end{aligned}$$

On the other side we have

$$\phi(x_j\cdot v^*, v)+\phi(v^*, x_j\cdot v)=-\sum_{i\in I}\langle x_j\cdot v^*|x_i\cdot v\rangle x_i-\sum_{i\in I}\langle v^*\cdot x_i|x_j\cdot v\rangle x_i$$

$$=-\sum_{i\in I}\langle v^*[x_i,x_j]v\rangle x_i=-\sum_{i\in I}\sum_{k\in I}\langle v^*x_kv\rangle C_{ij}^k x_i\,.$$

We now define a $\mathbb{Z}$-graded Lie algebra

$$\mathfrak{G}=\mathfrak{G}(\mathfrak{g}_0^e, V)=\sum_{n\in\mathbb{Z}}\mathfrak{G}_n\,, \tag{4.7}$$

where $\mathfrak{G}_0=\mathfrak{g}_0^e$, $\mathfrak{G}_{-1}=V$, $\mathfrak{G}_1=V^*$ and where $\mathfrak{G}^+=\sum_{n\geqq 1}\mathfrak{G}_n$ and $\mathfrak{G}^-=\sum_{n\leqq -1}\mathfrak{G}_n$ are the free Lie algebras generated by $\mathfrak{G}_1$ and $\mathfrak{G}_{-1}$, respectively. So for $n\geqq 1$, $\mathfrak{G}_n$ (respectively, $\mathfrak{G}_{-n}$) is the space spanned by all brackets of n vectors from $\mathfrak{G}_1$ (respectively, $\mathfrak{G}_{-1}$). The Lie brackets between $\mathfrak{G}_k$ and $\mathfrak{G}_{-\ell}$ for $k\geqq 1$, $\ell>1$ are defined inductively, that is,

$$[A,[B,C]]=[[A,B],C]+[B,[A,C]] \tag{4.8}$$

for $A\in\mathfrak{G}_k$, $[B,C]\in\mathfrak{G}_{-\ell}$, B, $C\in\mathfrak{G}^-$, and similarly for $k>1$, $\ell\geqq 1$. The brackets between $\mathfrak{G}_1$ and $\mathfrak{G}_{-1}$ are given by

$$[v^*,v]=\phi(v^*,v)\,, \tag{4.9}$$

and the brackets between $\mathfrak{G}_0$ and $\mathfrak{G}_1(\mathfrak{G}_{-1})$ are given by the $\mathfrak{g}_0^e$-module action,

$$[x,v]=x\cdot v \quad\text{and}\quad [x,v^*]=x\cdot v^*\,. \tag{4.10}$$

Proposition 4.1 shows that the Jacobi identity

$$[x,[v^*,v]]=[[x,v^*],v]+[v^*,[x,v]] \tag{4.11}$$

holds for all $x\in\mathfrak{G}_0$, $v^*\in\mathfrak{G}_1$, $v\in\mathfrak{G}_{-1}$. Note that $\mathfrak{G}_2$ (respectively, $\mathfrak{G}_{-2}$) and the exterior (wedge) product $V^*\wedge V^*$ (respectively, $V\wedge V$) are isomorphic as $\mathfrak{g}_0^e$-modules. The correspondence is just

$$[v_1^*,v_2^*] \;\leftrightarrow\; v_1^*\wedge v_2^* \text{ (respectively, } [v_1,v_2] \;\leftrightarrow\; v_1\wedge v_2)\,. \tag{4.12}$$

Define the subspaces (cf. [19])

$$\mathfrak{J}_k^+=\{x\in\mathfrak{G}_k|[y_1,[y_2,\ldots,[y_{k-1},x]\ldots]]=0 \quad\text{for all}\quad y_i\in\mathfrak{G}_{-1}\} \tag{4.13}$$

and

$$\mathfrak{J}_k^-=\{x\in\mathfrak{G}_{-k}|[y_1,[y_2,\ldots,[y_{k-1},x]\ldots]]=0 \quad\text{for all}\quad y_i\in\mathfrak{G}_1\} \tag{4.14}$$

for $k>1$. Let

$$\mathfrak{J}^+=\sum_{k>1}\mathfrak{J}_k^+,\mathfrak{J}^-=\sum_{k>1}\mathfrak{J}_k^- \quad\text{and}\quad \mathfrak{J}=\mathfrak{J}^++\mathfrak{J}^-\,. \tag{4.15}$$

Proposition 4.2. $\mathfrak{J}^+$, $\mathfrak{J}^-$, *and* $\mathfrak{J}$ *are ideals of* $\mathfrak{G}$.

Proof. Let $x\in\mathfrak{J}_k^+$ for $k>1$. If $y\in\mathfrak{G}_{-1}$ then it is obvious that $[x,y]\in\mathfrak{J}_{k-1}^+$. If $y\in\mathfrak{G}_1$ we wish to show that $[x,y]\in\mathfrak{J}_{k+1}^+$, that is, $[z_1, [z_2,\ldots[z_k, [x,y]]\ldots]]=0$ for all $z_i\in\mathfrak{G}_{-1}$. Written out using the Jacobi identity this expression is a summation of terms of the form $[[z_1, [z_2, \ldots [z_i, x]\ldots]], [z_{i+1}, [z_{i+2}, \ldots [z_k, y]\ldots]]]$. But since $[z_k, y] \in \mathfrak{G}_0$, $[z_{i+1}, [z_{i+2}, \ldots [z_k, y]\ldots]] \in \mathfrak{G}_{k-i-1}$, which may be distributed back into $[z_1, [z_2, \ldots [z_i, x]\ldots]]$ to yield a summation $\sum [w_1, [w_2, \ldots [w_{k-1}, x]\ldots]] = 0$ since $w_i \in \mathfrak{G}_{-1}$. The same argument shows that if $y \in \mathfrak{G}_0$ then $[x, y] \in \mathfrak{J}_k^+$. Since $\mathfrak{G}_1$, $\mathfrak{G}_0$, and $\mathfrak{G}_{-1}$ generate $\mathfrak{G}$, $\mathfrak{J}^+$ is an ideal of $\mathfrak{G}$. The argument for $\mathfrak{J}^-$ is symmetrical and so $\mathfrak{J} = \mathfrak{J}^+ + \mathfrak{J}^-$ is also an ideal of $\mathfrak{G}$.

Define the $\mathbb{Z}$-graded Lie algebra

$$\bar{\mathfrak{G}} = \bar{\mathfrak{G}}(\mathfrak{g}_0^e, V) = \mathfrak{G}/\mathfrak{J} = \mathfrak{G}^-/\mathfrak{J}^- + \mathfrak{G}_0 + \mathfrak{G}^+/\mathfrak{J}^+ \tag{4.16}$$

to be the canonical $\mathbb{Z}$-graded Lie algebra associated with $\mathfrak{g}_0^e$ and V. While our primary purpose is to show that $\bar{\mathfrak{G}}(\mathfrak{g}_0^e, V) \approx \mathfrak{F}$ it should be noted that there is no difficulty in generalizing the above construction to give $\bar{\mathfrak{G}}(\mathfrak{g}^{\wedge e}, V_0)$ for V_0 the basic module of any affine Kac-Moody algebra $\mathfrak{g}^\wedge$. The affine algebra $\mathfrak{g}^\wedge$ is constructed from a finite dimensional simple Lie algebra $\mathfrak{g}$ by the addition of a new simple root $\alpha_0 = -\tilde{\alpha}$ to $\{\alpha_1, \ldots, \alpha_\ell\}$, where $\tilde{\alpha}$ is the highest root of $\mathfrak{g}$. Then we may construct from $\mathfrak{g}^\wedge$ a Kac-Moody algebra $\mathfrak{g}^{\hat{\hat{}}}$ by the addition of a new simple root α_{-1} to $\{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$ such that α_{-1} is connected only to α_0 by just one line in the Dynkin diagram. The Cartan subalgebra $\mathfrak{h}^{\hat{\hat{}}} \subset \mathfrak{g}^{\hat{\hat{}}}$ will then be identified with the extended Cartan subalgebra $\mathfrak{h}^{\wedge e} \subset \mathfrak{g}^{\wedge e}$ and $\mathfrak{g}^{\hat{\hat{}}} \approx \bar{\mathfrak{G}}(\mathfrak{g}^{\wedge e}, V_0)$.

The simplest example of this construction is the algebra $\mathfrak{F}$ of type $A_1^{\hat{\hat{}}}$, which we studied first because of its Weyl group. The next cases one might consider are $A_2^{\hat{\hat{}}}$ and $B_2^{\hat{\hat{}}}$, whose Weyl groups are also quite interesting. The Weyl group of $B_2^{\hat{\hat{}}}$, for example, is isomorphic to the Klein-Fricke group Ψ_1^* (in Magnus' notation [34]) which contains the Picard group $\Psi_1 = \mathrm{PSL}_2(\mathbb{Z}[i])$ as a subgroup of index 4.

The algebra $\mathfrak{F}$ decomposes into $\mathfrak{F}_0^e$-modules

$$\ldots + \mathfrak{F}_{-2} + \mathfrak{F}_{-1} + \mathfrak{F}_0^e + \mathfrak{F}_1 + \mathfrak{F}_2 + \ldots, \tag{4.17}$$

where for $n \geqq 0$, $\mathfrak{F}_n$ (respectively, $\mathfrak{F}_{-n}$) is the space spanned by all brackets involving exactly n e_3's (respectively, f_3's).

Proposition 4.3. *We may identify $\mathfrak{F}_0^e = \mathfrak{G}_0$, $\mathfrak{F}_{-1} = \mathfrak{G}_{-1}$, and $\mathfrak{F}_1 = \mathfrak{G}_1$ in $\bar{\mathfrak{G}}(\mathfrak{g}_0^e, V)$ so that $v_0 \in V = \mathfrak{G}_{-1}$ corresponds to f_3, $v_0^* \in V^* = \mathfrak{G}_1$ corresponds to e_3, for $v \in V$ and $v^* \in V^*$ the bracket $[v^*, v]$ corresponds to $\phi(v^*, v)$, and $\mathfrak{g}_0^e$-module isomorphism $\mathfrak{F}_{-1} + \mathfrak{F}_0^e + \mathfrak{F}_1 \approx V + \mathfrak{g}_0^e + V^*$ respects those Lie brackets which stay within these graded pieces, provided that $\langle v_0^* | v_0 \rangle = 1$.*

Proof. From (2.2) we recall that

$$\begin{aligned} &[e_1, f_3] = 0, \quad [e_2, f_3] = 0, \\ &[h_1, f_3] = 0, \quad [h_2, f_3] = f_3, \quad [h_3, f_3] = -2f_3, \\ &[f_1, f_3] = 0, \quad [f_2[f_2, f_3]] = 0, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} &[f_1, e_3] = 0, \quad [f_2, e_3] = 0, \\ &[h_1, e_3] = 0, \quad [h_2, e_3] = -e_3, \quad [h_3, e_3] = 2e_3, \\ &[e_1, e_3] = 0, \quad [e_2[e_2, e_3]] = 0. \end{aligned} \tag{4.19}$$

It is easy to see that $\mathfrak{F}_{-1}$ (respectively, $\mathfrak{F}_1$) is generated as $\mathfrak{F}_0^e$-module by highest (respectively, lowest) weight vector f_3 (respectively, e_3) of weight $-\alpha_3$ (respectively, α_3), so $\mathfrak{F}_{-1}=V(\mathfrak{F}_1=V^*)$.

We have that $h_3=[e_3,f_3]$, which should correspond to

$$\phi(v_0^*,v_0)=-\sum_{i\in I}\langle v_0^* x_i v_0\rangle x_i$$

where $\{x_i|i\in I\}$ is any orthonormal basis of $\mathfrak{g}_0^e$. If we use the basis (3.17)–(3.19) then the only nonzero terms correspond to the basis vectors

$$\frac{1}{\sqrt{-2}}(c+d) \quad \text{and} \quad \frac{1}{\sqrt{2}}(c-d).$$

We have from (4.4) and (3.12) that

$$\begin{aligned}\phi(v_0^*,v_0)&=-\left\langle v_0^*\left(\frac{c+d}{\sqrt{-2}}\right)v_0\right\rangle\left(\frac{c+d}{\sqrt{-2}}\right)-\left\langle v_0^*\left(\frac{c-d}{\sqrt{2}}\right)v_0\right\rangle\left(\frac{c-d}{\sqrt{2}}\right)\\&=\langle v_0^* c v_0\rangle d+\langle v_0^* d v_0\rangle c\\&=\langle v_0^*|v_0\rangle(d-c)\\&=\langle v_0^*|v_0\rangle h_3,\end{aligned}$$

so the correspondence holds provided that $\langle v_0^*|v_0\rangle=1$. For arbitrary $v^*\in V^*$ and $v\in V$ we may assume that $v^*=v_0^*\cdot x_j$ and $v=x_k\cdot v_0$ for x_j,x_k basis vectors from an orthonormal basis $\{x_i|i\in I\}$ of $\mathfrak{g}_0^e$. Then $[[e_3,x_j],[x_k,f_3]]$ should correspond to $\phi(v_0^*\cdot x_j,x_k\cdot v_0)=\phi(v_0^*,x_j\cdot(x_k\cdot v_0))-[x_j,\phi(v_0^*,x_k\cdot v_0)]$ by Proposition 4.1. But from the Jacobi identity we have

$$[[e_3,x_j],[x_k,f_3]]=[e_3,[x_j,[x_k,f_3]]]-[x_j,[e_3,[x_k,f_3]]],$$

which shows that the correspondence holds.

Proposition 4.4. *The restriction of the invariant form* $(\cdot,\cdot)$ *of* $\mathfrak{F}$ *to* $\mathfrak{F}_{-1}+\mathfrak{F}_0^e+\mathfrak{F}_1$ *coincides with the invariant form* $\langle\cdot,\cdot\rangle$ *on* $\mathfrak{g}_0^e$ *and the pairing* $\langle\cdot|\cdot\rangle$ *of* V^* *and* V, *provided that* $\langle v_0^*|v_0\rangle=1$.

Proof. We see that $(e_3,f_3)=\frac{1}{2}([h_3,e_3],f_3)=\frac{1}{2}(h_3,[e_3,f_3])=\frac{1}{2}(h_3,h_3)=1$ agrees with $\langle v_0^*|v_0\rangle=1$. If $x_i,x_j\in\mathfrak{g}_0^e$ are from an orthonormal basis of $\mathfrak{g}_0^e$ we must show that $([x_i,e_3],[x_j,f_3])=\langle x_i\cdot v_0^*|x_j\cdot v_0\rangle$. From definition (4.6) we have

$$-\langle\phi(v^*,v),x_k\rangle=\langle v^* x_k v\rangle \tag{4.20}$$

for any $v^*\in V^*$, $v\in V$. From (3.12) and (3.16) it can be seen that the restriction of form $(\cdot,\cdot)$ to $\mathfrak{F}_0^e$ coincides with the form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}_0^e$. By invariance we have $([x_i,e_3],[x_j,f_3])=(x_i,[e_3,[x_j,f_3]])=\langle x_i,\phi(v_0^* x_j v_0)\rangle$ since $[e_3[x_j,f_3]]\in\mathfrak{g}_0^e$. But from (4.20) we have $\langle x_i,\phi(v_0^* x_j v_0)\rangle=-\langle v_0^* x_i|x_j v_0\rangle=\langle x_i\cdot v_0^*|x_j\cdot v_0\rangle$.

In order to see that $\mathfrak{F}\approx\mathfrak{G}(\mathfrak{g}_0^e,V)$ we must study more closely the ideal $\mathfrak{I}_2^-$. By definition

$$\begin{aligned}\mathfrak{I}_2^-&=\left\{\sum_k[v_1^k,v_2^k]\in\mathfrak{G}_{-2}\,\middle|\,\sum_k[v^*,[v_1^k,v_2^k]]=0\quad\text{for all}\quad v^*\in V^*=\mathfrak{G}_1\right\}\\&=\left\{\sum_k[v_1^k,v_2^k]\in\mathfrak{G}_{-2}\,\middle|\,\sum_k\langle v_1^*,[v_2^*,[v_1^k,v_2^k]]\rangle=0\quad\text{for all}\quad v_1^*,v_2^*\in V^*=\mathfrak{G}_1\right\}\end{aligned} \tag{4.21}$$

since V and V^* are dual. One has $\sum_k [v_1^k, v_2^k] \in \mathfrak{I}_2^-$ if and only if for all $v_1^*, v_2^* \in V^*$

$$\begin{aligned} 0 &= \sum_k (\langle v_1^*, [[v_2^*, v_1^k], v_2^k]\rangle + \langle v_1^*, [v_1^k, [v_2^*, v_2^k]]\rangle) \\ &= \sum_k \left(\left\langle v_1^*, -\sum_{i\in I} \langle v_2^* x_i v_1^k\rangle x_i \cdot v_2^k\right\rangle + \left\langle v_1^*, \sum_{i\in I} \langle v_2^* x_i v_2^k\rangle x_i \cdot v_1^k\right\rangle\right) \\ &= \sum_k \sum_{i\in I} (\langle v_1^* x_i v_1^k\rangle \langle v_2^* x_i v_2^k\rangle - \langle v_1^* x_i v_2^k\rangle \langle v_2^* x_i v_1^k\rangle), \end{aligned} \tag{4.22}$$

where $\{x_i | i\in I\}$ is any orthonormal basis of $\mathfrak{g}_0^e$.

Define a bilinear pairing of $\mathfrak{G}_{-2}$ with $\mathfrak{G}_2$ as follows. For any $v_1, v_2 \in V = \mathfrak{G}_{-1}$ and any $v_1^*, v_2^* \in V^* = \mathfrak{G}_1$ let

$$\langle [v_1^*, v_2^*], [v_1, v_2]\rangle = \langle v_1^* | v_1\rangle \langle v_2^* | v_2\rangle - \langle v_1^* | v_2\rangle \langle v_2^* | v_1\rangle. \tag{4.23}$$

Then each element of $\mathfrak{G}_2$ defines a linear functional on $\mathfrak{G}_{-2}$ and each element of $\mathfrak{G}_{-2}$ defines a linear functional on $\mathfrak{G}_2$. Note that

$$\langle [v_1^*, v_2^*], [x_i \cdot v_1, x_i \cdot v_2]\rangle = \langle v_1^* x_i v_1\rangle \langle v_2^* x_i v_2\rangle - \langle v_1^* x_i v_2\rangle \langle v_2^* x_i v_1\rangle \tag{4.24}$$

so if $\sum_k \sum_{i\in I} [x_i \cdot v_1^k, x_i \cdot v_2^k] = 0$ then $\sum [v_1^k, v_2^k] \in \mathfrak{I}_2^-$. To show the converse, supposing that

$$\sum_k \sum_{i\in I} [x_i \cdot v_1^k, x_i \cdot v_2^k] \neq 0 \tag{4.25}$$

one may use dual bases of V and V^* to find elements $v_{1m}^*, v_{2m}^* \in V^*$ such that

$$\left\langle \sum_m [v_{1m}^*, v_{2m}^*], \sum_k \sum_{i\in I} [x_i \cdot v_1^k, x_i \cdot v_2^k]\right\rangle \neq 0. \tag{4.26}$$

We have established

Lemma 4.5. *If $v_1^k, v_2^k \in V = \mathfrak{G}_{-1}$ and $\{x_i | i\in I\}$ is any orthonormal basis of $\mathfrak{G}_0 = \mathfrak{g}_0^e$, then $\sum_k [v_1^k, v_2^k] \in \mathfrak{I}_2^-$ if and only if $0 = \sum_k \sum_{i\in I} [x_i \cdot v_1^k, x_i \cdot v_2^k]$.*

Corollary 4.6. *We have $[v_0, [v_0, f_2]] \in \mathfrak{I}_2^-$.*

Proof. We must show $0 = \sum_{i\in I} [x_i \cdot v_0, x_i \cdot [v_0, f_2]]$. Using the orthonormal basis (3.17)–(3.19) and the formulas (4.4) we obtain after regrouping

$$\begin{aligned} &\sum_{0<k\in\mathbb{Z}} ([e(-k)\cdot v_0, f(k)\cdot [v^0, f_2]] + [f(-k)\cdot v_0, e(k)\cdot [v_0, f_2]] \\ &\quad + [h(-k)\cdot v_0, h(k)\cdot [v_0, f_2]]) - [c\cdot v_0, d\cdot [v_0, f_2]] - [d\cdot v_0, c\cdot [v_0, f_2]]. \end{aligned}$$

Since

$$c\cdot [v_0, f_2] = [c\cdot v_0, f_2] + [v_0, [c\cdot f_2]] = [v_0, f_2]$$

and

$$d\cdot [v_0, f_2] = [d\cdot v_0, f_2] + [v_0, [d, f_2]] = 0$$

the last two terms equal $[v_0, [v_0, f_2]]$. We have

$$f(k)\cdot[v_0, f_2] = [f(k)\cdot v_0, f_2] + [v_0, [f(k), f_2]]$$
$$= 0 + [v_0, -h(k-1) + k\delta_{k,1}c]$$

and

$$e(k)\cdot[v_0, f_2] = [e(k)\cdot v_0, f_2] + [v_0, [e(k), f_2]]$$
$$= 0 + [v_0, [e(k), e(-1)]] = 0$$

and

$$h(k)\cdot[v_0, f_2] = [h(k)\cdot v_0, f_2] + [v_0, [h(k), f_2]]$$
$$= 0 + [v_0, 2e(k-1)] = 0 \quad \text{for} \quad k > 0.$$

We finally obtain

$$[e(-1)\cdot v_0, [v_0, -h(0) + c]] + [v_0, [v_0, f_2]] = [f_2\cdot v_0, -v_0] + [v_0, [v_0, f_2]] = 0.$$

Theorem 4.7. $\mathfrak{F} \approx \bar{\mathfrak{G}}(\mathfrak{g}_0^e, V)$.

Proof. From Corollary 4.6 we have that $[v_0, [v_0, f_2]] \in \mathfrak{I}_2^-$ and from the symmetry between $\mathfrak{I}_2^-$ and $\mathfrak{I}_2^+$ we also have that $[v_0^*, [v_0^*, e_2]] \in \mathfrak{I}_2^+$. This shows that all of relations (2.2) hold in $\bar{\mathfrak{G}}(\mathfrak{g}_0^e, V)$, so by the simplicity of $\mathfrak{F}$ we have the isomorphism.

We would like to use Theorem 4.7 to say more about the root multiplicities of $\mathfrak{F}$. It is sufficient to understand the negative roots $\alpha = -n_1\alpha_1 - n_2\alpha_2 - n_3\alpha_3$, whose root space $\mathfrak{F}^\alpha$ is contained in $\mathfrak{F}_{-n_3}$. Recalling the notation of Proposition 2.1, we associate root α with $\nu(\alpha) = \begin{pmatrix} m_2 & m_1 \\ m_1 & m_3 \end{pmatrix} \in S_2(\mathbb{Z})$ and say that α is on level $m_3 = -n_3$. From Proposition 2.1 we know that α is real if and only if $\det(\nu(\alpha)) = -1$, in which case its multiplicity is 1. Any root β W-conjugate to an α on level 0 also has multiplicity 1. Aside from the real roots, any such root β is on the cone $\det(\nu(\beta)) = 0$. Because the ideal $\mathfrak{I}$ has trivial intersection with $\mathfrak{G}_1$ and $\mathfrak{G}_{-1}$ we have $\mathfrak{F}_1 \approx V^*$ with lowest weight α_3 and $\mathfrak{F}_{-1} \approx V$ with highest weight $-\alpha_3$. Thus, we have the following

Corollary 4.8. *Any root β W-conjugate to an α on level ± 1 has multiplicity $p(\det(\nu(\beta)) + 1)$ where p denotes the classical partition function.*

We would like to obtain a closed formula for the multiplicities of level ± 2 roots. First we will show that $\mathfrak{I}_2^-$ is the irreducible standard $\mathfrak{g}_0^e$-module with highest weight vector $[v_0, [v_0, f_2]]$ of weight $2\omega_1^0 - \omega_3 = -2\alpha_3 - \alpha_2$. Suppose that

$$v = \sum_{m=1}^{r} [v_1^m, v_2^m] \in \mathfrak{G}_{-2} \tag{4.27}$$

is a highest weight vector for $\mathfrak{g}_0^e$ of weight $\lambda = n_1\gamma_1^* + n_2\gamma_2^* + n_3\gamma_3^*$, so $n_2 = 2$ and $n_1 = 0$ or $n_1 = 2$. Using the orthonormal basis (3.17)–(3.19) and the result of Lemma 4.5 one may write out an expression $E(v)$ such that $v \in \mathfrak{I}_2^-$ if and only if $E(v) = 0$. If one uses (3.33) to write out $C\cdot v$ one obtains

$$C\cdot v=\sum_{m=1}^{r}[C\cdot v_1^m, v_2^m]+\sum_{m=1}^{r}[v_1^m, C\cdot v_2^m]+2E(v). \tag{4.28}$$

Thus we have $v\in\mathfrak{I}_2^-$ if and only if

$$C\cdot v=\sum_{m=1}^{r}([C\cdot v_1^m, v_2^m]+[v_1^m, C\cdot v_2^m]). \tag{4.29}$$

Since $v_1^m, v_2^m\in V^{-\alpha_3}$ and $-\alpha_3=\gamma_2^*-\gamma_3^*$ the scalar by which C acts on v_1^m and v_2^m is given by (3.34) to be zero. So $v\in\mathfrak{I}_2^-$ if and only if $C\cdot v=0$. From (3.84) and (3.85) this happens if and only if $\lambda=2\gamma_1^*+2\gamma_2^*-\gamma_3^*=2\omega_1^0-\omega_3=-2\alpha_3-\alpha_2$, so this is the only highest weight of a $\mathfrak{g}_0^e$-module in $\mathfrak{I}_2^-$. Corollary 4.6 shows that $[v_0,[v_0,f_2]]\in\mathfrak{I}_2^-$ is a vector of that weight.

We identify $\mathfrak{G}_{-2}$ with the wedge product $V\wedge V$ and with the antisymmetric tensors $A(V)$ in $V\otimes V$. This is quite natural since $v_1\wedge v_2=-v_2\wedge v_1$ and the action of $x\in\mathfrak{F}_0^e$ on $v_1\wedge v_2$ is given by

$$x\cdot(v_1\wedge v_2)=(x\cdot v_1)\wedge v_2+v_1\wedge(x\cdot v_2). \tag{4.30}$$

If $v_1, v_2\in\mathfrak{G}_{-1}=V$ then $[v_1, v_2]$ corresponds to $v_1\wedge v_2$ and the Jacobi identity

$$[x,[v_1,v_2]]=[[x,v_1],v_2]+[v_1,[x,v_2]] \tag{4.31}$$

corresponds to (4.30). The correspondence with $A(V)$ is given by

$$v_1\wedge v_2 \leftrightarrow v_1\otimes v_2-v_2\otimes v_1 \tag{4.32}$$

which is clearly a $\mathfrak{g}_0^e$-module isomorphism.

Therefore, $\mathfrak{F}_{-2}\approx\bar{\mathfrak{G}}_{-2}=\mathfrak{G}_{-2}/\mathfrak{I}_2^-$ is just $A(V)$ with the first standard module removed (see Corollary 3.5). If we denote the root multiplicity of α by $M(v(\alpha))=M\begin{pmatrix} m_2 & m_1\\ m_1 & m_3\end{pmatrix}$ then the character of $\mathfrak{F}_{-2}$ is

$$e^{2\gamma_2^*}\left[\left(\sum_{m\geq 0}M\begin{pmatrix} m & 0\\ 0 & 2\end{pmatrix}t^{2m}\right)\left(\sum_{k\in\mathbb{Z}}e^{2k\alpha_1}t^{4k^2}\right)+\left(\sum_{m\geq 0}M\begin{pmatrix} m & 1\\ 1 & 2\end{pmatrix}t^{2m-1}\right)\left(\sum_{k\in\mathbb{Z}}e^{(2k+1)\alpha_1}t^{(2k+1)^2}\right)\right], \tag{4.33}$$

where $t^2=q$. The principal specialization of this character is

$$\mathrm{ch}(\mathfrak{F}_{-2})=e^{2\gamma_2^*}\frac{\phi(u^4)^2}{\phi(u^2)}\sum_{n\geq 0}M(n-1)u^{n-1}, \tag{4.34}$$

where we have written more briefly

$$M(2m)=M\begin{pmatrix} m & 0\\ 0 & 2\end{pmatrix} \quad\text{and}\quad M(2m-1)=M\begin{pmatrix} m & 1\\ 1 & 2\end{pmatrix}. \tag{4.35}$$

Together with Corollary 3.5 this gives us the root multiplicities on the second level.

Corollary 4.9. *We have*

$$\sum_{n\geqq 0} M(n-1)u^n = \frac{\phi(u^2)}{\phi(u)\phi(u^4)}\frac{u^{-3}}{2}\left[\prod_{j\geqq 1}(1+u^{2j-1}) - \prod_{j\geqq 1}(1-u^{2j-1}) - 2u\right]$$

$$= \left[\sum_{n\geqq 0} p(n)u^n\right]\left[\prod_{j\geqq 1}(1-u^{4j-2})\right]$$

$$\cdot\frac{u^{-3}}{2}\left[\prod_{j\geqq 1}(1+u^{2j-1}) - \prod_{j\geqq 1}(1-u^{2j-1}) - 2u\right]$$

$$= \left[\sum_{n\geqq 0} p(n)u^n\right][1-u^{20}+u^{22}-u^{24}+u^{26}-2u^{28}+2u^{30}-2u^{32}+\ldots]. \tag{4.36}$$

It is quite remarkable that the first nonzero term beyond 1 in the above expansion is u^{20}. For that reason we have $M(n-1)=p(n)$ for $0\leqq n\leqq 19$, but $M(19)=p(20)-p(0)$.

We were led to our work on the root multiplicities of $\mathfrak{F}$ by computer generated data given to us by Moody. His program, based on the multiplicity formula in [36], gave data consistent with the formula

$$\dim(\mathfrak{F}^\alpha) = p(\det(N)+1), \tag{4.37}$$

where $v(\alpha)=N\in S_2(\mathbb{Z})$ from Sect. 2. This formula fails, however, for most roots not Weyl conjugate to roots from $\mathfrak{F}_{-1}+\mathfrak{F}_0^e+\mathfrak{F}_1$. A program which we developed computed multiplicities of some roots from $\mathfrak{F}_2$, $\mathfrak{F}_3$, and $\mathfrak{F}_4$. The smallest counterexample to (4.37) is when $\alpha=11\alpha_1+12\alpha_2+2\alpha_3$, corresponding to $M(19)$, for which $\dim(\mathfrak{F}^\alpha)=626$ instead of $p(20)=627$. We suspect that the root multiplicities of $\mathfrak{F}$ have some interesting number-theoretical meaning connected with ideal classes of imaginary quadratic fields. It is interesting to note that the roots α in $\mathfrak{F}_{-1}+\mathfrak{F}_0^e+\mathfrak{F}_1$ for which (4.37) holds correspond under v to binary forms associated with principal ideals.

Root multiplicities are also significant because they occur in the Weyl-Kac denominator formula for $\mathfrak{F}$.

Theorem 4.10. *The Weyl-Kac denominator formula for* $\mathfrak{F}$ *is*

$$\sum_{g\in PGL_2(\mathbb{Z})} \det(g)e^{2\pi i\operatorname{Tr}(gPg^t\mathfrak{Z})}$$

$$= e^{2\pi i\operatorname{Tr}(P\mathfrak{Z})}\prod_{0\leqq N\in S_2(\mathbb{Z})}(1-e^{2\pi i\operatorname{Tr}(N\mathfrak{Z})})^{\operatorname{Mult}(N)}\prod_{N\in v(R_W^-)}(1-e^{2\pi i\operatorname{Tr}(N\mathfrak{Z})}),$$

where $P=v(\varrho)=\begin{pmatrix}3 & 1/2\\ 1/2 & 2\end{pmatrix}$, $\mathfrak{Z}=\begin{pmatrix}z_3 & z_1\\ z_1 & z_2\end{pmatrix}$

$$v(R_W^-)=\left\{N=\begin{pmatrix}n_3 & n_1\\ n_1 & n_2\end{pmatrix}\in S_2(\mathbb{Z})|n_2n_3-n_1^2=-1, n_1\leqq n_2+n_3,\ 0\leqq n_2+n_3, 0\leqq n_2\right\}.$$

Proof. The Weyl-Kac denominator formula generally is

$$\sum_{w\in W}\det(w)e^{w\varrho-\varrho} = \prod_{\alpha\in R^-}(1-e^\alpha)^{m_\alpha}$$

where m_α is the dimension of the α root space. We have written this out for algebra $\mathfrak{F}$ as a function on the Siegel domain H_2 using the results of Sects. 2 and 3.

The exponents Mult(N) are the only difficult part of this identity since we have not given them all explicitly. They are, in fact, determined by the form of the identity, and Moody and Berman [36] have shown in general how to obtain a formula for root multiplicities from the denominator formula. Since our formula involves a summation over $PGL_2(\mathbb{Z})$ it may be of some interest to analysts. We challenge an analyst to prove this identity without the theory of Kac-Moody Lie algebras, or to shed more light on the meaning of the exponents Mult(N).

5. Theta Series and the Space of $SL_2(\mathbb{Z})$-Invariant $\mathfrak{F}_0^e$-Characters

The Weyl-Kac character formula for $\mathfrak{F}_0^e$ may be written as

$$\chi^\lambda = \frac{\sum_{w\in W_0} \det(w) e^{w(\lambda+\varrho_0)}}{\sum_{w\in W_0} \det(w) e^{w(\varrho_0)}}, \tag{5.1}$$

where $\varrho_0 = \omega_1^0 + \omega_2^0$. We will use the notation χ^λ for the standard $\mathfrak{F}_0^e$-character with dominant highest weight λ to distinguish it from the standard $\mathfrak{F}$-character X^λ. For such λ, $\chi^\lambda \in \mathscr{E}_0$. The dominant weights of $\mathfrak{F}_0^e$ are denoted by

$$P_0^{++} = \{n_1\gamma_1^* + n_2\gamma_2^* + t\gamma_3^* | t\in\mathbb{R}^+, n_1, n_2 \in \mathbb{Z}, n_2 \geqq n_1 \geqq 0\}. \tag{5.2}$$

The Weyl group W_0 decomposes into two cosets,

$$W_0^+ = \{(r_1r_2)^i | i\in\mathbb{Z}\} \tag{5.3}$$

and

$$W_0^- = \{(r_1r_2)^i r_1 | i\in\mathbb{Z}\}, \tag{5.4}$$

the even and odd elements, respectively. So we have

$$\begin{aligned}\sum_{w\in W_0} \det(w) e^{w(n_1\gamma_1^* + n_2\gamma_2^*)} &= \sum_{i\in\mathbb{Z}} (e^{T^i(n_1\gamma_1^* + n_2\gamma_2^*)} - e^{T^i(-n_1\gamma_1^* + n_2\gamma_2^*)}) \\ &= \sum_{i\in\mathbb{Z}} e^{(2in_2+n_1)\gamma_1^* + n_2\gamma_2^* + i(in_2+n_1)\gamma_3^*} \\ &\quad - \sum_{i\in\mathbb{Z}} e^{(2in_2-n_1)\gamma_1^* + n_2\gamma_2^* + i(in_2-n_1)\gamma_3^*}.\end{aligned} \tag{5.5}$$

Define the following formal theta series,

$$\Theta_{n,m} = e^{m\gamma_2^*} \sum_{j\in\mathbb{Z}+(n/2m)} e^{2jm\gamma_1^* + j^2 m\gamma_3^*} \tag{5.6}$$

for $m>0$ and $n\in\mathbb{R}$. If $0<m\in\mathbb{Z}$ and $n\in\mathbb{Z}$ then $\Theta_{n,m}\in\mathscr{E}$. Then one sees that (5.5) equals

$$e^{\left(\frac{-n_1^2}{4n_2}\right)\gamma_3^*}(\Theta_{n_1,n_2} - \Theta_{-n_1,n_2}). \tag{5.7}$$

Since

$$\lambda = m_1\omega_1^0 + m_2\omega_2^0 = m_1\gamma_1^* + (m_1+m_2)\gamma_2^* \tag{5.8}$$

and

$$\lambda + \varrho_0 = (m_1+1)\gamma_1^* + (m_1+m_2+2)\gamma_2^* \tag{5.9}$$

we have

$$\chi^\lambda = e^{-\left(\frac{(m_1+1)^2}{4(m_1+m_2+2)}-\frac{1}{8}\right)\gamma_3^*} \frac{(\Theta_{m_1+1, m_1+m_2+2} - \Theta_{-m_1-1, m_1+m_2+2})}{(\Theta_{1,2} - \Theta_{-1,2})}. \tag{5.10}$$

As was shown in Sect. 3, we may consider the formal theta series to be functions of three complex variables. Then we obtain from (3.92) and (5.6)

$$\Theta_{n,m}(z_1, z_2, z_3) = e^{2\pi i m z_2} \sum_{j \in \mathbb{Z} + (n/2m)} e^{4\pi i j m z_1 + 2\pi i j^2 m z_3}. \tag{5.11}$$

Lemma 5.1. *We have*

$$\Theta_{n,m}(z_1, z_2, z_3) = \frac{e^{2\pi i m\left(z_2 - \frac{z_1^2}{z_3}\right)}}{\sqrt{-2imz_3}} \sum_{j \in \frac{1}{2m}\mathbb{Z}} e^{2\pi i(-2jm(-z_1/z_3) + j^2 m(-1/z_3))} e^{2\pi i j n}$$

$$= \frac{1}{\sqrt{-2imz_3}} \sum_{\ell=-(m-1)}^{m} \Theta_{\ell,m}\left(\frac{-z_1}{z_3}, z_2 - \frac{z_1^2}{z_3}, \frac{-1}{z_3}\right) e^{-\pi i n \ell/m}.$$

Proof. We use the Poisson summation formula which says that if $f(x)$ is a continous function of a real variable x and if

$$\hat{f}(y) = \int_{\mathbb{R}} e^{-2\pi i x y} f(x) dx \tag{5.12}$$

then

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n). \tag{5.13}$$

If we define

$$f(x) = e^{4\pi i m x z_1 + 2\pi i (m x^2 + n x) z_3} \tag{5.14}$$

for $x \in \mathbb{R}$ then we find from (5.11) that

$$\Theta_{n,m}(z_1, 0, z_3) = e^{2\pi i n z_1 + 2\pi i (n^2/4m) z_3} \sum_{j \in \mathbb{Z}} f(j). \tag{5.15}$$

If we let $x_0 = (-y + 2mz_1 + nz_3)/(2mz_3)$ then

$$\hat{f}(y) = e^{-2\pi i m z_3 x_0^2} \int_{\mathbb{R}} e^{2\pi i m z_3 (x + x_0)^2} dx \tag{5.16}$$

$$= \frac{1}{\sqrt{-2imz_3}} e^{-2\pi i m z_3 x_0^2} \tag{5.17}$$

since

$$\int_{\mathbb{R}} e^{-pu^2} du = \sqrt{(\pi/p)}.$$

After substitution of the expression for x_0 into (5.17) and some algebraic manipulation, the first part of the lemma can be obtained.

For the second part, note that

$$\Theta_{-\ell,m}\left(\frac{-z_1}{z_3}, z_2-\frac{z_1^2}{z_3}, \frac{-1}{z_3}\right)$$
$$=e^{2\pi im(z_2-z_1^2/z_3)}\sum_{k\in\mathbb{Z}-(\ell/2m)} e^{4\pi ikm(-z_1/z_3)+2\pi ik^2m(-1/z_3)} \tag{5.18}$$

and that

$$\frac{1}{2m}\mathbb{Z}=\bigcup_{\ell=-(m-1)}^{m}\left\{\mathbb{Z}+\frac{\ell}{2m}\right\}. \tag{5.19}$$

So using the first part we can write

$$\Theta_{n,m}(z_1,z_2,z_3)$$
$$=\frac{e^{2\pi im(z_2-z_1^2/z_3)}}{\sqrt{-2imz_3}}\sum_{\ell=-m}^{m-1}\sum_{k\in\mathbb{Z}+(\ell/2m)} e^{-4\pi imk(-z_1/z_3)+2\pi imk^2(-1/z_3)}e^{2\pi ink}.$$

By changing k to $-k$ and by using (5.18) we obtain the result.

Definition. Let $J(z_1,z_2,z_3)=\Theta_{1,2}(z_1,z_2,z_3)-\Theta_{-1,2}(z_1,z_2,z_3)$.

Corollary 5.2. $J(z_1,z_2,z_3)=\dfrac{-i}{\sqrt{-iz_3}}J\left(\dfrac{-z_1}{z_3}, z_2-\dfrac{z_1^2}{z_3}, \dfrac{-1}{z_3}\right)$.

Remark. Looijenga [26] has introduced the formal theta series for arbitrary affine Kac-Moody algebras and proved the analog of Corollary 5.2 in that generality.

Definition. If $\lambda=m_1\omega_1^0+m_2\omega_2^0+m_3\gamma_3^*=m_1\gamma_1^*+(m_1+m_2)\gamma_2^*+m_3\gamma_3^*\in\mathfrak{h}^*$ then we call λ a weight of level m_1+m_2.

If $V\in\mathscr{C}$ is an $\mathfrak{F}$-module then it is also an $\mathfrak{F}_0^e$-module ($\mathscr{C}\subseteq\mathscr{C}_0$) and decomposes into the direct sum of standard $\mathfrak{F}_0^e$-modules, each of which has character $\chi^\lambda=e^{m_3\gamma_3^*}\chi^{m_1\omega_1^0+m_2\omega_2^0}\in\mathscr{E}_0$ of level $M=m_1+m_2\geqq 0$.
Since each such λ is in P^+ we can write

$$X(V)=\sum_{M\geqq 0}\sum_{\substack{m_1+m_2=M\\ m_1,m_2\in\mathbb{Z}^+}}\chi^{m_1\omega_1^0+m_2\omega_2^0}\sum_{m_3\in\mathbb{Z}^+}A(m_1,m_2,m_3)e^{m_3\gamma_3^*}, \tag{5.20}$$

where $A(m_1,m_2,m_3)$ is the multiplicity of χ^λ in $X(V)$.

Definition. Let the level of $\mathfrak{F}$-module $V\in\mathscr{C}$ be the least positive integer M such that $A(m_1,m_2,m_3)\neq 0$ for some m_1,m_2,m_3 such that $m_1+m_2=M$.

The level of V is just the smallest level any weight λ such that χ^λ occurs in the decomposition of $X(V)$. If V^λ is the standard $\mathfrak{F}$-module with dominant highest weight

$$\lambda=n_1\omega_1+n_2\omega_2+n_3\omega_3=n_1\gamma_1^*+(n_1+n_2)\gamma_2^*+(n_1+n_2+n_3)\gamma_3^*$$

then all weights of V^λ are of the form

$$\lambda-p_1\alpha_1-p_2\alpha_2-p_3\alpha_3=\lambda-2(p_1-p_2)\gamma_1^*+p_3\gamma_2^*+(p_2-p_3)\gamma_3^*$$

for

$$p_1, p_2, p_3\in\mathbb{Z}^+.$$

So the level of any such weight is $m_1+m_2+p_3 \geqq m_1+m_2$ and the level of any $\mathfrak{F}_0^e$-module occuring in the decomposition of V^λ must be at least m_1+m_2. The vectors of weight λ generate an $\mathfrak{F}_0^e$-module of level m_1+m_2 in V^λ, so the level of V^λ is exactly m_1+m_2, the level of λ.

If we decompose the character of $\mathfrak{F}$-module $V \in \mathscr{C}$ of level M_0 into standard $\mathfrak{F}_0^e$-characters we get

$$X(V)=\sum_{M \geqq M_0} \sum_{j=0}^{M} \chi^{j\omega_1^0+(M-j)\omega_2^0} \sum_{m \in \mathbb{Z}^+} A(j, M-j, m) e^{m\gamma_3^*}. \tag{5.21}$$

Definition. An $\mathfrak{F}_0^e$-module in $\mathscr{C}_0$ is said to be of level M of each of its weights is of level M.

Note that the action of $\mathfrak{F}_0^e$ on any weight can only move it by a linear combination of α_1 and α_2 (γ_1^* and γ_3^*) so the γ_2^* coefficient, which equals the level, is unchanged. Thus, the weights of any standard $\mathfrak{F}_0^e$-module lie entirely on one level, and an $\mathfrak{F}_0^e$-module in $\mathscr{C}_0$ is of level M if and only if all of its irreducible standard $\mathfrak{F}_0^e$-components have highest weights of level M. There are $M+1$ isomorphism classes of such standard $\mathfrak{F}_0^e$-modules, representative characters being

$$\{\chi^{j\omega_1^0+(M-j)\omega_2^0} | 0 \leqq j \leqq M, \ j \in \mathbb{Z}\}. \tag{5.22}$$

Given a standard $\mathfrak{F}$-module V^λ, $\lambda \in P^{++}$, of level M, we have a unique standard $\mathfrak{F}_0^e$-module v^λ of level M whose highest weight is λ. In fact, v^λ is exactly the lowest level $\mathfrak{F}_0^e$-component of V^λ.

Conversely, given a standard $\mathfrak{F}_0^e$-module v^λ of level M with $\lambda \in P^{++}$ (note the restriction on the γ_3^* coefficient) there is a unique standard $\mathfrak{F}$-module, V^λ, whose lowest level $\mathfrak{F}_0^e$-component is v^λ.

It is easy to perform this lifting and descent for sums of standard characters. We will see later the significance of this correspondence for the theory of Siegel modular forms.

Definition. We shall call an irreducible standard $\mathfrak{F}_0^e$-module and its character $\mathfrak{F}$-dominant whenever the highest weight of the module is in P^{++}. We shall call an arbitrary $\mathfrak{F}_0^e$-module and its character $\mathfrak{F}$-dominant whenever each irreducible standard component is $\mathfrak{F}$-dominant.

We shall denote by $\mathscr{C}'_0$ the subcategory of $\mathscr{C}_0$ consisting of those $\mathfrak{F}_0^e$-modules which are $\mathfrak{F}$-dominant. The above discussion makes clear the following.

Proposition 5.3. *There is a natural one-to-one correspondence between the category of $\mathfrak{F}$-modules $\mathscr{C}$ and the category of $\mathfrak{F}$-dominant $\mathfrak{F}_0^e$-modules $\mathscr{C}'_0$, which induces a one-to-one correspondence between $\mathfrak{F}$-characters in $\mathscr{E}$ and $\mathfrak{F}$-dominant $\mathfrak{F}_0^e$-characters in $\mathscr{E}_0$.*

Remark. Given a level M $\mathfrak{F}$-dominant $\mathfrak{F}_0^e$-character, Proposition 5.3 tells us that there exist $\mathfrak{F}_0^e$-characters on the levels above M naturally associated with it. In theory we have operators which produce these higher level characters from the one on level M. If the level M character is standard irreducible then these operators construct the corresponding standard irreducible $\mathfrak{F}$-character. Specific formulas for such operators would be very interesting. For the case of level $M=1$ they might be related to the Hecke operators which will be discussed later in Sects. 6 and 7.

Consider an arbitrary level M $\mathfrak{F}_0^e$-character of the form

$$\chi_M = \sum_{j=0}^{M} \chi^{j\omega_1^0 + (M-j)\omega_2^0} \sum_{m\in\mathbb{Z}^+} A_j(m) e^{m\gamma_3^*}. \tag{5.23}$$

We understand from (3.92) that χ_M may be viewed as a function of z_1, z_2, z_3. Using (5.10) we can write

$$\begin{aligned}\chi_M(z_1,z_2,z_3) &= \sum_{j=0}^{M} \chi^{j\omega_1^0+(M-j)\omega_2^0}(z_1,z_2,z_3) \sum_{m\geqq 0} A_j(m) e^{2\pi i m z_3} \\ &= \sum_{j=0}^{M} \bar{\chi}^{j\omega_1^0+(M-j)\omega_2^0}(z_1,z_2,z_3) c_j(z_3),\end{aligned} \tag{5.24}$$

where

$$\bar{\chi}^{j\omega_1+(M-j)\omega_2^0}(z_1,z_2,z_3) = \frac{\Theta_{j+1,M+2}(z_1,z_2,z_3) - \Theta_{-j-1,M+2}(z_1,z_2,z_3)}{\Theta_{1,2}(z_1,z_2,z_3) - \Theta_{-1,2}(z_1,z_2,z_3)} \tag{5.25}$$

and

$$c_j(z_3) = e^{-\left(\frac{(j+1)^2}{4(M+2)} - \frac{1}{8}\right)2\pi i z_3} \sum_{m\geqq 0} A_j(m) e^{2\pi i m z_3}. \tag{5.26}$$

Proposition 5.4. *Let $\chi_M(z_1,z_2,z_3)$ be an $\mathfrak{F}_0^e$-character of level M written in the form of* (5.24). *Let $k\in\mathbb{C}$. Then*

$$\chi_M(z_1,z_2,z_3) = (-z_3)^{-k} \chi_M\left(\frac{-z_1}{z_3}, z_2 - \frac{z_1^2}{z_3}, \frac{-1}{z_3}\right)$$

if and only if

$$c_n\left(\frac{-1}{z_3}\right)(-z_3)^{-k} = \sqrt{\frac{2}{M+2}} \sum_{j=0}^{M} c_j(z_3) \sin\left(\frac{\pi(j+1)(n+1)}{M+2}\right) \quad \textit{for} \quad 0 \leqq n \leqq M.$$

Proof. For $n \in \frac{1}{2(M+2)}\mathbb{Z}$ define

$$c_n'(z_3) = \begin{cases} c_{m-1}(z_3) & \text{if } 2(M+2)n \equiv m (\operatorname{mod} 2(M+2)), 1 \leqq m \leqq M+1 \\ -c_{m-1}(z_3) & \text{if } 2(M+2)n \equiv -m (\operatorname{mod} 2(M+2)), 1 \leqq m \leqq M+1 \\ 0 & \text{if } 2(M+2)n \equiv 0, M+2 (\operatorname{mod} 2(M+2)). \end{cases}$$

We have

$$\chi_M(z_1,z_2,z_3) = \frac{\sum_{j=0}^{M} e^{2\pi i (M+2) z_2} (S_j^+ - S_j^-) c_j(z_3)}{J(z_1,z_2,z_3)}, \tag{5.27}$$

where

$$S_j^\pm = \sum_{r\in\mathbb{Z}\pm\frac{j+1}{2(M+2)}} e^{4\pi i r (M+2) z_1 + 2\pi i r^2 (M+2) z_3}.$$

This can be rewritten as

$$\frac{e^{2\pi i(M+2)z_2}\displaystyle\sum_{r\in\frac{\mathbb{Z}}{2(M+2)}} c'_r(z_3)e^{4\pi i r(M+2)z_1+2\pi i r^2(M+2)z_3}}{J(z_1,z_2,z_3)} \tag{5.28}$$

because $\mathbb{Z}\pm\dfrac{j+1}{2(M+2)}\subseteq\dfrac{\mathbb{Z}}{2(M+2)}=\left\{p+\dfrac{\ell}{2(M+2)}\,\middle|\,p\in\mathbb{Z},\ -(M+1)\leqq\ell\leqq M+2\right\}$ so that if $r=p\pm\dfrac{j+1}{2(M+2)}$ then the corresponding term in the summation is multiplied by $c_j(z_3)$, $-c_j(z_3)$ or 0 according to the following rules. We have

$$2(M+2)r=2(M+2)p\pm(j+1)$$

with $1\leqq j+1\leqq M+1$ so $\ell=j+1$. If $2(M+2)r\equiv\ell(\mathrm{mod}\,2(M+2))$ for $1\leqq\ell\leqq M+1$ then we want $c_j(z_3)=c_{\ell-1}(z_3)=c'_r(z_3)$. If $2(M+2)r\equiv-\ell(\mathrm{mod}\,2(M+2))$ for $1\leqq\ell\leqq M+1$ then we want $-c_j(z_3)=-c_{\ell-1}(z_3)=c'_r(z_3)$. The only other possibilities for $2(M+2)r\,(\mathrm{mod}\,2(M+2))$ are 0 and $M+2$, for which we want $0=c'_r(z_3)$.

By using Lemma 5.1 on the theta functions in the numerator of (5.25) and Corollary 5.2 on the denominator, we can obtain the following expression for $\chi_M(z_1,z_2,z_3)$:

$$\frac{\dfrac{e^{2\pi i(M+2)(z_2-z_1^2/z_3)}}{\sqrt{-2(M+2)iz_3}}\displaystyle\sum_r\left[\sum_{j=0}^{M}(e^{2\pi i r(j+1)}-e^{-2\pi i r(j+1)})c_j(z_3)\right]e^{-4\pi i r(M+2)\frac{-z_1}{z_3}+2\pi i r^2(M+2)\frac{-1}{z_3}}}{\dfrac{-i}{\sqrt{-iz_3}}J\left(\dfrac{-z_1}{z_3},\,z_2-\dfrac{z_1^2}{z_3},\,\dfrac{-1}{z_3}\right)}, \tag{5.29}$$

where the summations over r are taken over $\dfrac{\mathbb{Z}}{2(M+2)}$.

It is easy to check that

$$\sum_{j=0}^{M}(e^{2\pi i r(j+1)}-e^{-2\pi i r(j+1)})c_j(z_3)=\sum_{\ell=0}^{2M+3}c'_{\left(\frac{\ell}{2(M+2)}\right)}(z_3)e^{2\pi i\ell r}. \tag{5.30}$$

Let (5.31) be the expression obtained from (5.29) by substitution of (5.30), and let (5.32) be the expression for $\chi_M\left(\dfrac{-z_1}{z_3},\,z_2-\dfrac{z_1^2}{z_3},\,\dfrac{-1}{z_3}\right)$ obtained by substitution of variables in (5.28). After changing r to $-r$ in (5.32) and using $c'_{-r}=-c'_r$ one sees that $\chi_M(z_1,z_2,z_3)=(-z_3)^{-k}\chi_M\left(\dfrac{-z_1}{z_3},\,z_2-\dfrac{z_1^2}{z_3},\,\dfrac{-1}{z_3}\right)$ holds if and only if

$$\frac{1}{i\sqrt{2(M+2)}}\sum_{\ell=-(M+1)}^{M+1}c'_{\left(\frac{\ell}{2(M+2)}\right)}(z_3)e^{2\pi i\frac{\ell n}{2(M+2)}}=(-z_3)^{-k}c'_{\left(\frac{n}{2(M+2)}\right)}\left(\frac{-1}{z_3}\right)\quad\text{for}\quad 1\leqq n\leqq 2(M+2). \tag{5.33}$$

Both sides (5.33) can be seen to be 0 when $n=0$ or $M+2$ and if (5.33) holds for some n, $1\leqq n\leqq M+1$, then it also holds for $-n$, so the condition (5.33) can be considered just for $1 \leqq n \leqq M+1$. Writing out (5.33) in terms of $c_j(z_3)$ and $c_n\left(\frac{-1}{z_3}\right)$ we finally get

$$\sum_{j=0}^{M} c_j(z_3)\left(\frac{e^{2\pi i\frac{(j+1)(n+1)}{2(M+2)}} - e^{-2\pi i\frac{(j+1)(n+1)}{2(M+2)}}}{i\sqrt{2(M+2)}}\right) = (-z_3)^{-k} c_n\left(\frac{-1}{z_3}\right)$$

$$= \sqrt{\frac{2}{M+2}} \sum_{j=0}^{M} c_j(z_3) \sin\left(\frac{\pi(j+1)(n+1)}{M+2}\right) \quad \text{for} \quad 0 \leqq n \leqq M.$$

Definition. Let $\mathscr{M}_k$ be the complex vector space spanned by those $\mathfrak{F}_0^e$-characters of the form

$$\chi(z_1, z_2, z_3) = \sum_{M \geqq 0} \chi_M(z_1, z_2, z_3) \tag{5.34}$$

with $\chi_M(z_1, z_2, z_3)$ satisfying the conditions of Proposition 5.4. Then $\mathscr{M}_k$ is naturally graded according to the level of its corresponding modules:

$$\mathscr{M}_k = \sum_{M \geqq 0} \mathscr{M}_k(M), \tag{5.35}$$

where $\mathscr{M}_k(M)$ is the space spanned by the characters of level M modules satisfying Proposition 5.4.

By definition any $\chi(z_1, z_2, z_3) \in \mathscr{M}_k$ can be written in the form

$$\sum A(m_1, m_2, m_3) e^{2\pi i(m_1 z_1 + m_2 z_2 + m_3 z_3)}, \tag{5.36}$$

where the sum is taken over integers m_1, m_2, m_3 such that $4m_2 m_3 \geqq m_1^2$, and $m_2, m_3 \geqq 0$. So any such χ remains invariant under the translations $z_j \to z_j + 1$, $j = 1, 2, 3$. From the formulas $\chi(z_1, z_2, z_3 + 1) = \chi(z_1, z_2, z_3)$ and

$$\chi(z_1, z_2, z_3) = (-z_3)^{-k} \chi\left(\frac{-z_1}{z_3}, z_2 - \frac{z_1^2}{z_3}, \frac{-1}{z_3}\right)$$

one may compute that for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\chi(z_1, z_2, z_3) = (cz_3 + d)^{-k} \chi\left(\frac{z_1}{cz_3 + d}, z_2 - \frac{cz_1^2}{cz_3 + d}, \frac{az_3 + b}{cz_3 + d}\right) \tag{5.37}$$

because $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ generate $\mathrm{SL}_2(\mathbb{Z})$.

Remark. Naturally we are interested in the dimension of the space $\mathscr{M}_k(M)$. The answer to this problem, surprisingly, is connected with the theory of genus 2 Siegel modular forms which will be discussed in Sect. 7.

Now let us find the form of characters in $\mathscr{M}_k(0)$ and $\mathscr{M}_k(1)$. From (5.24), since $\bar{\chi}^{0\omega_1^\circ + 0\omega_2^\circ} = 1$, we have for an arbitrary $\chi_0(z_1, z_2, z_3)$ of level 0,

$$\chi_0(z_1, z_2, z_3) = c_0(z_3) = \sum_{m \geqq 0} A_0(m) e^{2\pi i m t} \in \mathscr{M}_k(0) \tag{5.38}$$

if and only if it satisfies

$$(cz_3+d)^{-k}c_0\left(\frac{az_3+b}{cz_3+d}\right)=c_0(z_3) \quad \text{for any} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{5.39}$$

If we denote for $h=0,1$ the classical Jacobi theta functions by

$$\Theta_h(z_1,z_3)=\sum_{j\in\mathbb{Z}} e^{2\pi i\left((2j+h)z_1+\left(j+\frac{h}{2}\right)^2 z_3\right)} \tag{5.40}$$

then

$$\Theta_{h,1}(z_1,z_2,z_3)=e^{2\pi i z_2}\Theta_h(z_1,z_3), \tag{5.41}$$

$$\bar{\chi}^{\omega_1^0}(z_1,z_2,z_3)=e^{2\pi i z_2}\eta^{-1}(e^{2\pi i z_3})\Theta_1(z_1,z_3), \tag{5.42}$$

and

$$\bar{\chi}^{\omega_2^0}(z_1,z_2,z_3)=e^{2\pi i z_2}\eta^{-1}(e^{2\pi i z_3})\Theta_0(z_1,z_3), \tag{5.43}$$

where $\eta(X)=X^{1/24}\phi(X)$ denotes the Dedekind η-function. The conditions for

$$\chi(z_1,z_2,z_3)=\bar{\chi}^{\omega_2^0}(z_1,z_2,z_3)c_0(z_3)+\bar{\chi}^{\omega_1}(z_1,z_2,z_3)c_1(z_3) \tag{5.44}$$

to be in $\mathcal{M}_k(1)$ are two equations which may be stated in vector notation:

$$(-z_3)^{-k}\begin{pmatrix} c_0\left(\frac{-1}{z_3}\right) \\ c_1\left(\frac{-1}{z_3}\right) \end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} c_0(z_3) \\ c_1(z_3) \end{pmatrix}. \tag{5.45}$$

This particular vectorial modular form has occurred in the investigations of Maass [30–32] into Siegel modular forms of genus 2 in connection with the "lifting" of elliptic modular forms. In order to make the connection between the theory of algebra $\mathfrak{F}$ and the theory of Siegel modular forms we must now review some classical definitions and results.

6. Siegel Modular Forms and Hecke Operators

We recall some classical facts about Siegel modular forms of genus $n \geqq 1$ which can be found in [1, 13, 40].

Definition. The Siegel domain (upper half-plane) of genus $n \geqq 1$ is the set of all symmetric $n \times n$ complex matrices having positive definite imaginary part:

$$H_n=\{\mathfrak{Z}=X+iY\in S_n(\mathbb{C})\,|\,Y>0\}. \tag{6.1}$$

Definition. The real symplectic group of genus n is

$$\mathrm{Sp}_n(\mathbb{R})=\{N\in \mathrm{GL}_{2n}(\mathbb{R})\,|\,NJN^t=J\},$$

where J is the $2n \times 2n$ matrix whose $n \times n$ blocks are

$$\begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}. \tag{6.2}$$

The group $\mathrm{Sp}_n(\mathbb{R})$ acts on H_n by the analytic automorphism

$$N\cdot\mathfrak{Z}=(A\mathfrak{Z}+B)(C\mathfrak{Z}+D)^{-1} \tag{6.3}$$

for $N=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in \mathrm{Sp}_n(\mathbb{R})$ and $\mathfrak{Z}\in H_n$.

An important discrete subgroup of $\mathrm{Sp}_n(\mathbb{R})$ is the subgroup of integral symplectic matrices;

$$\mathrm{Sp}_n(\mathbb{Z})=\mathrm{Sp}_n(\mathbb{R})\cap M_{2n}(\mathbb{Z}) \tag{6.4}$$

which is called the *Siegel modular group of genus n*. One sometimes denotes it by Γ_n.

A *Siegel modular form* of genus n and weight k (natural number) is any function $f(\mathfrak{Z})$ which is holomorphic on H_n and satisfies the two conditions,

(I) For every $N=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in \mathrm{Sp}_n(\mathbb{Z})$ and every $\mathfrak{Z}\in H_n$, $\det(C\mathfrak{Z}+D)^{-k}f(N\cdot\mathfrak{Z})$ $=f(\mathfrak{Z})$, in which case we say f is invariant under N,

(II) Function $f(\mathfrak{Z})$ is bounded in every domain of the form

$$\{\mathfrak{Z}=X+iY\in H_n \mid Y\geqq cI_n, c>0\}.$$

For $n>1$ Condition (II) is known to follow from (I) and the holomorphism of f. The set of all Siegel modular forms of genus n and weight k forms a finite dimensional complex vector space, denoted $\mathfrak{M}_k^n$.

Every $f\in\mathfrak{M}_k^n$ has a Fourier expansion

$$f(\mathfrak{Z})=\sum_{N\geqq 0} a(N)e^{2\pi i\,\mathrm{Tr}(N\mathfrak{Z})}, \tag{6.5}$$

where N runs over the positive semidefinite matrices in the set

$$S_n'(\mathbb{Z})=\{N=(n_{ij})\in M_n(\mathbb{Q}) \mid N=N^t, t_{ii}, 2t_{ij}\in\mathbb{Z} \quad\text{for}\quad i\neq j\}. \tag{6.6}$$

If we apply Condition (I) with an element $\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}\in\Gamma_n$ we get

$$\det(A)^k f(A\mathfrak{Z}A^t)=f(\mathfrak{Z}) \quad\text{for any}\quad A\in \mathrm{GL}_n(\mathbb{Z})$$

which implies the following relation for the Fourier coefficients of f;

$$a(ANA^t)=\det(A)^k a(N). \tag{6.7}$$

If all coefficients $a(N)$ such that $\det(N)=0$ vanish, then form $f(\mathfrak{Z})$ is called a *parabolic form*. We denote the space of all parabolic forms of genus n and weight k by $\mathfrak{N}_k^n$.

A very important class of examples of modular forms of genus n and weight k consists of the Eisenstein series, defined as

$$\Psi_k(\mathfrak{Z})=\sum_{C,D}\det(C\mathfrak{Z}+D)^{-k}, \tag{6.8}$$

where the summation extends over all inequivalent bottom blocks of elements of $\mathrm{Sp}_n(\mathbb{Z})$ with respect to left multiplications by matrices in $\mathrm{SL}_n(\mathbb{Z})$.

A classical theorem of H. Braun shows that the series $\Psi_k(\mathfrak{Z})$ is absolutely convergent for $k>n+1$ and is a modular form of weight k. The notation of (6.8) will be reserved for the Eisenstein series of genus 2 and we will write ϕ_k for the case of genus 1.

If we let $g_2=60\phi_4$, $g_3=140\phi_6$, $\Delta=g_2^3-27g_3^2$ then for even k one has

$$\begin{aligned}\mathfrak{M}_k^1&=\Delta\mathfrak{M}_{k-12}^1\oplus\mathbb{C}\phi_k\\&=\mathfrak{N}_k^1\oplus\mathbb{C}\phi_k.\end{aligned}\tag{6.9}$$

It is well known that ϕ_4 and ϕ_6 are algebraically independent over $\mathbb{C}$,

$$\sum_{k=0}^{\infty}\mathfrak{M}_k^1=\mathbb{C}[\phi_4,\phi_6],\tag{6.10}$$

and that,

$$\dim\mathfrak{M}_k^1=\text{cardinality of the set }\{(a,b)\in(\mathbb{Z}^+)^2|4a+6b=k\}.\tag{6.11}$$

An analogous result for genus 2 has been obtained by Igusa [13].

Theorem 6.1 (Igusa). *The genus 2 Eisenstein series Ψ_4, Ψ_6, Ψ_{10}, Ψ_{12} are algebraically independent over $\mathbb{C}$ and*

$$\sum_{k=0}^{\infty}\mathfrak{M}_k^2=\mathbb{C}[\Psi_4,\Psi_6,\Psi_{10},\Psi_{12}].$$

Remark (see [13]). Since $\phi_4\phi_6=\phi_{10}$ and $3^2\cdot7^2\phi_4^3+2\cdot5^3\phi_6^2=691\phi_{12}$ we can define normalized parabolic forms

$$\chi_{10}=-43876\cdot2^{-12}\cdot3^{-5}\cdot5^{-2}\cdot7^{-1}\cdot53^{-1}(\Psi_4\Psi_6-\Psi_{10})$$

and

$$\chi_{12}=131\cdot593\cdot2^{-13}\cdot3^{-7}\cdot5^{-3}\cdot7^{-2}\cdot337^{-1}(3^2\cdot7^2\Psi_4^3+2\cdot5^3\Psi_6^2-691\Psi_{12}).\tag{6.12}$$

Then we can write

$$\sum_{k=0}^{\infty}\mathfrak{M}_k^2=\mathbb{C}[\Psi_4,\Psi_6,\chi_{10},\chi_{12}].\tag{6.13}$$

Corollary 6.2 $\dim\mathfrak{M}_k^2=$ *Cardinality of the set*

$$\{(a,b,c,d)\in(\mathbb{Z}^+)^4|k=4a+6b+10c+12d\}.$$

The Siegel operator is a map $\Phi:\mathfrak{M}_k^n\to\mathfrak{M}_k^{n-1}$ defined for $n>1$ as follows: Let $\mathfrak{Z}'\in H_{n-1}$ and $0<\lambda\in\mathbb{R}$, then $\begin{pmatrix}\mathfrak{Z}' & 0\\ 0 & i\lambda\end{pmatrix}\in H_n$, the limit

$$(\Phi f)(\mathfrak{Z}')=\lim_{\lambda\to\infty}f\left(\begin{pmatrix}\mathfrak{Z}' & 0\\ 0 & i\lambda\end{pmatrix}\right)\tag{6.14}$$

exists for $f\in\mathfrak{M}_k^n$, and $\Phi f\in\mathfrak{M}_k^{n-1}$. One can see that the parabolic forms $\mathfrak{N}_k^n$ are just the kernel of Φ. In the case of genus $n=2$, if f is written as the Fourier series (6.5) then the only terms of f which survive the limit process of (6.14) are those corresponding to $N=\begin{pmatrix} n_3 & 0 \\ 0 & 0 \end{pmatrix}$, so if we let $a'(n_3)=a(N)$ and $z'=z_3$ then

$$(\Phi f)(z_3)=\sum_{n_3\geqq 0} a'(n_3)e^{2\pi i n_3 z_3}.$$

A very important class of operators on the space of modular forms is named for Hecke, who considered them first for genus 1. We will discuss and generalize that case here.

Let

$$\Delta_1(m)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in M_2(\mathbb{Z})|ad-bc=m\right\} \tag{6.15}$$

for $1\leqq m\in\mathbb{Z}$. Then $\Gamma_1=\mathrm{SL}_2(\mathbb{Z})$ acts on $\Delta_1(m)$ by left multiplication and $\Delta_1(m)$ breaks up into finitely many orbits under that action. A complete set of representatives, one from each orbit, may be taken to be

$$\bar{\Delta}_1(m)=\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\in M_2(\mathbb{Z})|ad=m, a\geqq 1, 0\leqq b<d\right\}. \tag{6.16}$$

If $f(z)\in\mathfrak{M}_k^1$, define the operator $T_k(m)$ by

$$(T_k(m)f)(z)=m^{k-1}\sum_{\sigma\in\Gamma_1\backslash\bar{\Delta}_1(m)} f(\sigma\cdot z)(cz+d)^{-k} \quad \text{for} \quad \sigma=\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\in\Gamma_1\backslash\bar{\Delta}_1(m). \tag{6.17}$$

Remark. From (5.38) and (5.39) we see that $\mathfrak{M}_k^1$ coincides with $\mathcal{M}_k(0)$.

We will now extend the action of $T_k(m)$ to all of $\mathcal{M}_k$.

Definition. If $\chi(z_1, z_2, z_3)\in\mathcal{M}_k$ define

$$\begin{aligned}(T_k(m)\chi)(z_1, z_2, z_3) &= m^{k-1}\sum_{\sigma\in\Gamma_1\backslash\bar{\Delta}_1(m)} \chi\left(\frac{mz_1}{cz_3+d}, m\left(z_2-\frac{cz_1^2}{cz_3+d}\right), \frac{az_3+b}{cz_3+d}\right)(cz_3+d)^{-k}\\ \text{for} \quad \sigma&=\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\in\Gamma_1\backslash\bar{\Delta}_1(m), \quad 1\leqq m\in\mathbb{Z}, \quad 0<k\in\mathbb{Z}.\end{aligned} \tag{6.18}$$

Clearly $T_k(1)=$ Identity operator.

We will abbreviate the expression under the summation by the slash operator:

$$(\chi|_k g)(z_1, z_2, z_3)=\chi\left(\frac{mz_1}{cz_3+d}, m\left(z_2-\frac{cz_1^2}{cz_3+d}\right), \frac{az_3+b}{cz_3+d}\right)(cz_3+d)^{-k}$$
$$\text{for} \quad g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Delta_1(m). \tag{6.19}$$

It is straightforward to verify that for any $g_1\in\Delta_1(m)$, $g_2\in\Delta_1(n)$ we have

$$(\chi|_k g_1)|_k g_2=(\chi|_k(g_1 g_2)). \tag{6.20}$$

The condition that $\mathfrak{F}_0^e$-character $\chi\in\mathcal{M}_k$ is exactly that

$$\chi|_k g=\chi \quad \text{for all} \quad g\in\Delta_1(1)=\mathrm{SL}_2(\mathbb{Z})=\Gamma_1. \tag{6.21}$$

So

$$(T_k(m)\chi)(z_1,z_2,z_3)=m^{k-1}\sum_{\sigma\in\Gamma_1\backslash\bar{\Delta}_1(m)}(\chi|_k\sigma)(z_1,z_2,z_3) \tag{6.22}$$

is clearly well-defined on $\mathcal{M}_k$. It is also clear that $T_k(m)\chi\in\mathcal{M}_k$ because if $\{\sigma_1,\ldots,\sigma_r\}$ is a complete set of representatives of $\Gamma_1\backslash\bar{\Delta}_1(m)$ then so is $\{\sigma_1 g,\ldots,\sigma_r g\}$ for any $g\in\Gamma_1$.

In fact, $T_k(m):\mathcal{M}_k(M)\to\mathcal{M}_k(mM)$ can be seen as follows. The level of an $\mathfrak{F}_0^e$-character $\chi(z_1,z_2,z_3)$ is determined by the exponential factor involving z_2 since that determines the γ_2^* coefficient of each weight in the corresponding $\mathfrak{F}_0^e$-module. If $\chi\in\mathcal{M}_k(M)$ we can write

$$\chi(z_1,z_2,z_3)=e^{2\pi iMz_2}\sum a(n_1,M,n_3)e^{2\pi i(n_1z_1+n_3z_3)} \tag{6.23}$$

and using the representatives (6.16)

$$(T_k(m)\chi)(z_1,z_2,z_3)=m^{k-1}\sum_{\substack{a\geqq 1\\ ad=m}}\sum_{0\leqq b<d}d^{-k}\chi\left(az_1,mz_2,\frac{az_3+b}{d}\right) \tag{6.24}$$

in which the exponential involving z_2 is $e^{2\pi imMz_2}$. So $T_k(m)\chi\in\mathcal{M}_k(mM)$.

7. The Correspondence of $\mathfrak{F}$-Characters with Siegel Modular Forms of Genus 2

We have seen in Sects. 3 and 5 how $\mathfrak{F}$-characters $X(z_1,z_2,z_3)\in\mathcal{E}$ and $\mathfrak{F}_0^e$-characters $\chi(z_1,z_2,z_3)\in\mathcal{E}_0$ can be viewed as functions on the genus 2 Siegel domain H_2. We wish to investigate the transformation invariance of these functions and determine when they are genus 2 Siegel modular forms.

Write an $\mathfrak{F}$-character $X\in\mathcal{E}$ as

$$X(\mathfrak{z})=X(z_1,z_2,z_3)=\sum_{\substack{n_2n_3\geqq n_1^2\\ n_2,n_3\geqq 0}}a(n_1,n_2,n_3)e^{2\pi i(n_1z_1+n_2z_2+n_3z_3)}$$

$$\text{for}\quad \mathfrak{z}=\begin{pmatrix} z_3 & z_1\\ z_1 & z_2\end{pmatrix}\in H_2\,. \tag{7.1}$$

The action of the Weyl group $W\approx\mathrm{PGL}_2(\mathbb{Z})$ on H_2, given by (3.102), determines an action of $g^t\in\mathrm{PGL}_2(\mathbb{Z})$ on functions of $\mathfrak{z}\in H_2$;

$$g^t\cdot f(\mathfrak{z})=f(g\mathfrak{z}g^t)\,. \tag{7.2}$$

But $g^t\cdot X(\mathfrak{z})=X(g\mathfrak{z}g^t)=X(\mathfrak{z})$ because of (3.91), the invariance of the form $(\cdot,\cdot)$ on $S_2(\mathbb{C})$ under $\mathrm{PGL}_2(\mathbb{Z})$, and the fact that weights of $\mathfrak{F}$-modules which are W-conjugate have equal multiplicities. It is also clear from (7.1) that for any $M\in S_2(\mathbb{Z})$, $X(\mathfrak{z}+M)=X(\mathfrak{z})$. Since the root lattice Q is isomorphic to $S_2(\mathbb{Z})$ we may consider the action of $q\in Q$ on $X(\mathfrak{z})$ to be $q\cdot X(\mathfrak{z})=X(\mathfrak{z}+M)$ for $M=\nu(q)\in S_2(\mathbb{Z})$.

The Weyl group W acts on Q, so we may form the semidirect product

$$W\ltimes Q=\{w,q)\in W\times Q|(w_1,q_1)\cdot(w_2,q_2)=(w_1w_2,w_2^{-1}\cdot q_1+q_2)\}\,. \tag{7.3}$$

Proposition 7.1. *For $q\in Q$, let $v(q)=M\in S_2(\mathbb{Z})$ and for $w\in W$, let $\bar{v}(w)=x$ determine $x\in \mathrm{PGL}_2(\mathbb{Z})$. Then the map $\mathfrak{f}: W\ltimes Q\to \mathrm{Sp}_2(\mathbb{Z})/\pm I$ defined by $\mathfrak{f}(w,q)=\begin{pmatrix} x & xM \\ 0 & (x^t)^{-1}\end{pmatrix}$ is an isomorphism of $W\ltimes Q$ with the "upper triangular" subgroup in the projective symplectic group.*

Definition. Let the action of $W\ltimes Q$ on a function $f(\mathfrak{z})$ be given by

$$(w,q)\cdot f(\mathfrak{z})=\mathfrak{f}(w,q)\cdot f(\mathfrak{z}). \tag{7.4}$$

The action of $\mathrm{Sp}_2(\mathbb{Z})$ on H_2 given in (6.3) is well-defined on the projective symplectic group $\mathrm{Sp}_2(\mathbb{Z})/\pm I$ whose elements we still denote by $\begin{pmatrix} A & B \\ C & D\end{pmatrix}$. If $N=\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1}\end{pmatrix}$ then $N\cdot\mathfrak{z}=A\mathfrak{z}A^t$ and $\det((A^t)^{-1})^{-k}X(A\mathfrak{z}A^t)=\det(A)^k X(\mathfrak{z})$. Since $A\in\mathrm{PGL}_2(\mathbb{Z})$, Condition (I) of the definition of Siegel modular form will be satisfied for all such A if and only if k is even. If $N=\begin{pmatrix} I & M \\ 0 & I\end{pmatrix}$ for $M\in S_2(\mathbb{Z})$ then $N\cdot\mathfrak{z}=\mathfrak{z}+M$ and $\det(I)^{-k}X(\mathfrak{z}+M)=X(\mathfrak{z})$. So we have

Proposition 7.2. *Let $X(\mathfrak{z})$ be any $\mathfrak{F}$-character. Then for any $N=\begin{pmatrix} A & B \\ 0 & D\end{pmatrix}\in\mathrm{Sp}_2(\mathbb{Z})$, and any $\mathfrak{z}\in H_2$, $X(\mathfrak{z})=\det(D)^{-k}X(N\cdot\mathfrak{z})=X(N\cdot\mathfrak{z})$ if k is even.*

Proposition 7.3. *Let $f(\mathfrak{z})\in\mathfrak{M}_k^2$. Then there exists a unique collection of standard $\mathfrak{F}$-characters $\{X^\lambda(\mathfrak{z})\in\mathscr{E}\,|\,\lambda\in P^{++}\}$ and a unique collection of complex numbers $\{c_\lambda\in\mathbb{C}\,|\,\lambda\in P^{++}\}$ such that $f(\mathfrak{z})=\sum_{\lambda\in P^{++}} c_\lambda X^\lambda(\mathfrak{z})$.*

Proof. By Igusa's result (Theorem 6.1) we may assume $k\geqq 4$ is even. So if $N=\begin{pmatrix} A & B \\ 0 & D\end{pmatrix}\in\mathrm{Sp}_2(\mathbb{Z})$ we have $f(N\cdot\mathfrak{z})=f(\mathfrak{z})$ and $X(N\cdot\mathfrak{z})=X(\mathfrak{z})$ for any $X(\mathfrak{z})\in\mathscr{E}$.

Recall the Weyl-Kac character formula (Theorem 3.1). The numerator and denominator may be considered as functions on H_2 since $\lambda+\varrho\in P^{++}$. Let us use the notation $\mathscr{A}^{\lambda+\varrho}(\mathfrak{z})$ for the numerator, so the denominator is just $\mathscr{A}^{\varrho}(\mathfrak{z})$. It is easy to see that if $A\in\mathrm{PGL}_2(\mathbb{Z})$ and $M\in S_2(\mathbb{Z})$ then

$$\mathscr{A}^{\lambda+\varrho}(A\mathfrak{z}A^t)=\det(A)\mathscr{A}^{\lambda+\varrho}(\mathfrak{z}) \quad\text{and}\quad \mathscr{A}^{\lambda+\varrho}(\mathfrak{z}+M)=\mathscr{A}^{\lambda+\varrho}(\mathfrak{z}). \tag{7.5}$$

In fact, we can write out the Fourier expansion

$$\mathscr{A}^{\lambda+\varrho}(\mathfrak{z})=\sum_{A\in\mathrm{PGL}_2(\mathbb{Z})}\det(A)e^{2\pi i\,\mathrm{Tr}(ANA^t\mathfrak{z})}, \tag{7.6}$$

where $0<N\in S_2'(\mathbb{Z})$ corresponds to $\lambda+\varrho\in P^{++}$. In the theory of Lie algebras one defines the set of strictly dominant weights to be

$$P^{+++}=\{n_1\omega_1+n_2\omega_2+n_3\omega_3\,|\,0<n_i\in\mathbb{Z}\quad\text{for}\quad 1\leqq i\leqq 3\}. \tag{7.7}$$

Then

$$P^{+++}=\{n_1\gamma_1^*+n_2\gamma_2^*+n_3\gamma_3^*\in P^+\,|\,n_3>n_2>n_1>0\}, \tag{7.8}$$

$$v(W\cdot P^{+++})=\{N\in S_2'(\mathbb{Z})\,|\,N>0\}, \tag{7.9}$$

each $\mathrm{PGL}_2(\mathbb{Z})$ orbit in $v(W \cdot P^{+++})$ has a unique representative in $v(P^{+++})$, and $P^{+++} = \varrho + P^{++}$. If $\lambda \in P^{++}$ then $\lambda + \varrho \in P^{+++}$, and conversely.

Consider the function

$$f'(\mathfrak{z}) = f(\mathfrak{z})\mathscr{A}^{\varrho}(\mathfrak{z}). \tag{7.10}$$

Certainly $f'(\mathfrak{z}+M) = f'(\mathfrak{z})$ for any $M \in S_2(\mathbb{Z})$ and $f'(A\mathfrak{z}A^t) = \det(A) f'(\mathfrak{z})$ for any $A \in \mathrm{PGL}_2(\mathbb{Z})$. Because each term of $\mathscr{A}^{\varrho}(\mathfrak{z})$ corresponds to some $0 < N \in S_2'(\mathbb{Z})$ each nonzero term of $f'(\mathfrak{z})$ corresponds to some $0 < N_1 \in S_2'(\mathbb{Z})$. If we decompose $f'(\mathfrak{z})$ into $\mathrm{PGL}_2(\mathbb{Z})$ orbits we may write

$$f'(\mathfrak{z}) = \sum_{N \in v(P^{+++})} a(N) \sum_{A \in \mathrm{PGL}_2(\mathbb{Z})} \det(A) e^{2\pi i \operatorname{Tr}(ANA^t \mathfrak{z})}, \tag{7.11}$$

where $a(N)$ is the Fourier coefficient of $f'(\mathfrak{z})$. Since $N \in v(P^{+++})$ there is a unique $\lambda \in P^{++}$ such that $v(\lambda + \varrho) = N$ and we recognize the inner sum of (7.11) as $\mathscr{A}^{\lambda+\varrho}(\mathfrak{z})$. Dividing by $\mathscr{A}^{\varrho}(\mathfrak{z})$ and using the character formula $\mathscr{A}^{\lambda+\varrho}(\mathfrak{z})/\mathscr{A}^{\varrho}(\mathfrak{z}) = X^{\lambda}(\mathfrak{z})$ we have $f(\mathfrak{z}) = \sum_{\lambda \in P^{++}} c_\lambda X^\lambda(\mathfrak{z})$ where $c_\lambda = a(N)$ for $N = v(\lambda + \varrho)$.

Remark. In the proof of Proposition 7.3 we have used only the convergence of $f(\mathfrak{z})$ and its invariance under the parabolic subgroup.

Following Maass [31] we consider two subgroups of $\mathrm{Sp}_2(\mathbb{Z})$.

Definition. Let

$$E_1 = \left\{ \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ \hline c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\} \tag{7.12}$$

and

$$E_2 = \left\{ \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ \hline 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{array}\right) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\}. \tag{7.13}$$

It is clear that E_1 and E_2 are subgroups of $\mathrm{Sp}_2(\mathbb{Z})$ isomorphic to $\mathrm{SL}_2(\mathbb{Z})$.

Lemma 7.4. *The subgroups E_1, E_2 and $\left\{\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{Z}) \middle| B = B^t \right\}$ suffice to generate* $\mathrm{Sp}_2(\mathbb{Z})$.

Proof. This well-known fact is an easy exercise.

Proposition 7.5. *Necessary and sufficient conditions for $f(\mathfrak{z}) \in \mathfrak{M}_k^2$ are:*

(a) $f\left(\dfrac{z_1}{cz_3+d}, z_2 - \dfrac{cz_1^2}{cz_3+d}, \dfrac{az_3+b}{cz_3+d}\right)(cz_3+d)^{-k} = f(z_1, z_2, z_3)$,

(b) $f(z_1+m_1, z_2+m_2, z_3+m_3) = f(z_1, z_2, z_3)$, *and*

(c) $f(z_1, z_2, z_3) = f(z_1, z_3, z_2)$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *and all* $m_1, m_2, m_3 \in \mathbb{Z}$.

Proof. Condition (a) is equivalent to invariance of $f(\mathfrak{Z})$ under E_1, (b) is trivial, and (c) implies a symmetry between E_1 and E_2 so that Lemma 7.4 gives the result.

Lemma 7.6. *Write the Fourier series of a function $f(\mathfrak{Z})=f(z_1,z_2,z_3)$ on H_2 satisfying* (b) *as*

$$f(z_1,z_2,z_3)=\sum_{n_2\in\mathbb{Z}^+} f_{n_2}(z_1,z_2,z_3), \tag{7.14}$$

where

$$f_{n_2}(z_1,z_2,z_3)=e^{2\pi i n_2 z_2}\sum_{\substack{n_3\geqq 0\\ 4n_2n_3\geqq n_1^2}} a(n_1,n_2,n_3)e^{2\pi i(n_1z_1+n_3z_3)}. \tag{7.15}$$

Then $f(z_1,z_2,z_3)$ satisfies (a) *if and only if $f_{n_2}(z_1,z_2,z_3)$ satisfies* (a) *for each $n_2\geqq 0$.*

Proof. If each f_{n_2} satisfies (a) then certainly their sum f does also. The condition (a) for $f(z_1,z_2,z_3)$ is just that for each

$$\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}), f(z_1,z_2,z_3)=(f|_k\sigma)(z_1,z_2,z_3)=\sum_{n_2\geqq 0}(f_{n_2}|_k\sigma)(z_1,z_2,z_3).$$

So $\sum_{n_2\geqq 0} f_{n_2}(z_1,z_2,z_3)=\sum_{n_2\geqq 0}(f_{n_2}|_k\sigma)(z_1,z_2,z_3)$ and comparing the expansions of both sides with respect to the variable z_2, we get $f_{n_2}(z_1,z_2,z_3)=(f_{n_2}|_k\sigma)(z_1,z_2,z_3)$ for each $n_2\geqq 0$.

Theorem 7.7. *Let $f(\mathfrak{Z})\in\mathfrak{M}_k^2$. Then there is a unique expression*

$$f(\mathfrak{Z})=\sum_{M\geqq 0}\chi_M(z_1,z_2,z_3), \tag{7.16}$$

where $\chi_M\in\mathcal{M}_k(M)$ is a level M $\mathfrak{F}_0^e$-character satisfying (a) *and* (b). *Thus, $\mathfrak{M}_k^2\subseteqq\mathcal{M}_k$.*

Proof. By Proposition 7.3 we can uniquely write $f(\mathfrak{Z})=\sum_{\lambda\in P^{++}} c_\lambda X^\lambda(\mathfrak{Z})$. Each standard $\mathfrak{F}$-character $X^\lambda(\mathfrak{Z})$ can be uniquely decomposed into $\mathfrak{F}_0^e$-characters according to level as in (5.20). The resulting decomposition of $f(\mathfrak{Z})$ is that given in (7.14), so for each $M\geqq 0$ $f_M(z_1,z_2,z_3)=\chi_M(z_1,z_2,z_3)$ is an $\mathfrak{F}_0^e$-character of level M. Since $f(\mathfrak{Z})\in\mathfrak{M}_k^2$, $f(\mathfrak{Z})$ satisfies (a) and (b), so each $\chi_M(z_1,z_2,z_3)$ satisfies (a) and (b), by Lemma 7.6. That is, $\chi_M\in\mathcal{M}_k(M)$ and $f(\mathfrak{Z})\in\mathcal{M}_k$, so $\mathfrak{M}_k^2\subset\mathcal{M}_k$.

Corollary 7.8. $\mathfrak{M}_k^2=\{f(\mathfrak{Z})\in\mathcal{M}_k | f(z_1,z_2,z_3)=f(z_1,z_3,z_2)\}$.

Proof. Clear from Proposition 7.5, Lemma 7.6, and Theorem 7.7.

Definition. Let the level of a function $f(\mathfrak{Z})$ satisfying (b) be the least positive integer n_2 such that $f_{n_2}\neq 0$ in (7.14).

Definition. Let $\mathfrak{T}_k^2(m)$ be the subspace in $\mathfrak{M}_k^2$ of functions whose level is at least m. Then we have the filtration of $\mathfrak{M}_k^2$

$$\mathfrak{M}_k^2=\mathfrak{T}_k^2(0)\supset\mathfrak{T}_k^2(1)\supset\mathfrak{T}_k^2(2)\supset\ldots\supset\mathfrak{T}_k^2(m)\supset\ldots \tag{7.17}$$

and it is clear that if $f(\mathfrak{Z})\in\mathfrak{T}_{k_1}^2(m_1)$ and $g(\mathfrak{Z})\in\mathfrak{T}_{k_2}^2(m_2)$ then

$$f(\mathfrak{Z})g(\mathfrak{Z})\in\mathfrak{T}_{k_1+k_2}^2(m_1+m_2).$$

Also note that because of condition (c) $\mathfrak{T}_k^2(1)$ is exactly the subspace $\mathfrak{N}_k^2$ of parabolic forms.

Recall from the remarks following Theorem 6.1 that

$$\sum_{k\geqq 0}\mathfrak{M}_k^2=\mathbb{C}[\Psi_4,\Psi_6,\chi_{10},\chi_{12}]$$

is the Igusa ring generated by the Eisenstein series Ψ_4,Ψ_6 and the parabolic forms χ_{10},χ_{12}. According to the above definition, Ψ_4 and Ψ_6 are of level 0, χ_{10}, and χ_{12} are of level 1. So the monomial $\Psi_4^a\Psi_6^b\chi_{10}^c\chi_{12}^d$ for $a,b,c,d\in\mathbb{Z}^+$ is in $\mathfrak{T}_k^2(m)$ but not in $\mathfrak{T}_k^2(m+1)$, where $m=c+d$ and $k=4a+6b+10c+12d$. We may write

$$\mathbb{C}[\Psi_4,\Psi_6,\chi_{10},\chi_{12}]=\sum_{k\geqq 0}\sum_{m\geqq 0}\mathfrak{M}_k^2(m), \tag{7.18}$$

where

$$\mathfrak{M}_k^2(m)=\sum_{\substack{a,b,c,d\in\mathbb{Z}^+\\ 4a+6b+10c+12d=k\\ c+d=m}}\mathbb{C}\Psi_4^a\Psi_6^b\chi_{10}^c\chi_{12}^d. \tag{7.19}$$

Then

$$\mathfrak{M}_k^2=\sum_{m\geqq 0}\mathfrak{M}_k^2(m) \tag{7.20}$$

and we have for each $m\geqq 0$ a vector space isomorphism

$$\mathfrak{M}_k^2(m)\approx\mathfrak{T}_k^2(m)/\mathfrak{T}_k^2(m+1). \tag{7.21}$$

From (7.19) and the linear independence of the monomials which span $\mathfrak{M}_k^2(m)$ it is clear that the dimension of $\mathfrak{M}_k^2(m)$ over $\mathbb{C}$ equals the cardinality of the set $S(k,m)$ defined to be

$$\{(a,b,c,d)\in(\mathbb{Z}^+)^4 \mid k=4a+6b+10c+12d, c+d=m\}. \tag{7.22}$$

We wish to study the relationship between the spaces $\mathfrak{M}_k^2(m)$ and $\mathcal{M}_k(m)$ for each $m\geqq 0$. It is obvious that $\mathfrak{M}_k^2(0)\approx\mathfrak{M}_k^1\approx\mathcal{M}_k(0)$ [see (5.38), (5.39)], where the first isomorphism is given by the Siegel operator Φ.

Definition. For each $m\geqq 0$ let the linear transformation $L_m:\mathfrak{M}_k^2\to\mathcal{M}_k$ be defined by $L_m(f(\mathfrak{Z}))=f_m(\mathfrak{Z})$. So L_0 is just the Siegel operator Φ.

Definition. Let $\mathcal{M}_k'$ [resp., $\mathcal{M}_k'(m)$] be the subspace of $\mathcal{M}_k$ [resp., $\mathcal{M}_k(m)$] consisting of those $PSL_2(\mathbb{Z})$-invariant $\mathfrak{F}_0^e$-characters which are $\mathfrak{F}$-dominant. Recall that an irreducible standard $\mathfrak{F}_0^e$-module and its character are called $\mathfrak{F}$-dominant if the highest weight is in P^{++} (2.20). and that an arbitrary $\mathfrak{F}_0^e$-module and its character are called $\mathfrak{F}$-dominant if each of its irreducible standard components is $\mathfrak{F}$-dominant. One sees that

$$\mathcal{M}_k'(m)=\{f_m(\mathfrak{Z})\in\mathcal{M}_k(m) \mid A(n_1,m,n_3)=0 \text{ for all } n_3<m\}, \tag{7.23}$$

where

$$f_m(\mathfrak{Z})=e^{2\pi i m z_2}\sum_{\substack{4mn_3\geqq n_1^2\\ n_3\geqq 0}}A(n_1,m,n_3)e^{2\pi i(n_1z_1+n_3z_3)}. \tag{7.24}$$

Because of condition (c) we see that if $f(\mathfrak{Z})\in\mathfrak{T}_k^2(m)$ then $L_m(f(\mathfrak{Z}))\in\mathcal{M}_k'(m)$. Since $\mathfrak{M}_k^2(m)\subset\mathfrak{T}_k^2(m)$, we have

$$L_m':\mathfrak{M}_k^2(m)\to\mathcal{M}_k'(m)\,, \tag{7.25}$$

where we have denoted by L_m' the restriction of L_m to the subspace $\mathfrak{M}_k^2(m)$. We can now find the dimensions of the spaces $\mathcal{M}_k'(m)$.

Theorem 7.9. *The linear transformation $L_m':\mathfrak{M}_k^2(m)\to\mathcal{M}_k'(m)$ is an isomorphism for each $m\geqq 0$.*

Proof. We have the basis of $\mathfrak{M}_k^2(m)$

$$\{\Psi_4^a\Psi_6^b\chi_{10}^c\chi_{12}^d\,|\,(a,b,c,d)\in S(k,m)\}\,. \tag{7.26}$$

Let $(\Psi_4)_0=L_0(\Psi_4)$, $(\Psi_6)_0=L_0(\Psi_6)$, $(\chi_{10})_1=L_1(\chi_{10})$, and $(\chi_{12})_1=L_1(\chi_{12})$. Then it is clear that

$$\begin{aligned}L_m(\Psi_4^a\Psi_6^b\chi_{10}^c\chi_{12}^d)&=(\Psi_4)_0^a(\Psi_6)_0^b(\chi_{10})_1^c(\chi_{12})_1^d\\&=\varphi_4^a\varphi_6^b(\chi_{10})_1^c(\chi_{12})_1^d\,,\end{aligned} \tag{7.27}$$

where φ_4 and φ_6 are the genus 1 Eisenstein series involving only z_3. Also, $(\chi_{10})_1\in\mathcal{M}_{10}'(1)$ and $(\chi_{12})_1\in\mathcal{M}_{12}'(1)$ are $\mathfrak{F}_0^e$-characters which may be written [see (5.23)] as:

$$(\chi_{10})_1(\mathfrak{Z})=D_{10}^1(z_3)\chi^{\omega_1^0}(\mathfrak{Z})+D_{10}^2(z_3)\chi^{\omega_2^0}(\mathfrak{Z})\,, \tag{7.28}$$

$$(\chi_{12})_1(\mathfrak{Z})=D_{12}^1(z_3)\chi^{\omega_1^0}(\mathfrak{Z})+D_{12}^2(z_3)\chi^{\omega_2^0}(\mathfrak{Z})\,, \tag{7.29}$$

where the coefficients $D_k^j(z_3)$, $k=10,12$, $j=1,2$ are power series in $q=e^{2\pi i z_3}$ with constant term 0.

We claim that this system of equations can be inverted, expressing $\chi^{\omega_1^0}$ and $\chi^{\omega_2^0}$ as a combination of $(\chi_{10})_1$ and $(\chi_{12})_1$ with coefficients being Laurent series in q.

Let us write, for $k=10,12$, $N=\begin{pmatrix}n_3 & n_1/2\\ n_1/2 & n_2\end{pmatrix}$, $\mathfrak{Z}=\begin{pmatrix}z_3 & z_1\\ z_1 & z_2\end{pmatrix}$,

$$\chi_k(\mathfrak{Z})=\sum_{N>0}c_k(N)e^{2\pi i\,\mathrm{Tr}(N\mathfrak{Z})} \tag{7.30}$$

and

$$(\chi_k)_1(\mathfrak{Z})=e^{2\pi i z_2}\sum_{n_3\geqq 1}e^{2\pi i n_3 z_3}\left[c_k\begin{pmatrix}n_3 & 0\\ 0 & 1\end{pmatrix}\Theta_0(z_1,z_3)+e^{-2\pi i z_3/4}c_k\begin{pmatrix}n_3 & 1/2\\ 1/2 & 1\end{pmatrix}\Theta_1(z_1,z_3)\right], \tag{7.31}$$

where $\Theta_h(z_1,z_3)$, $h=0,1$, is given in (5.40).

Let

$$\mathscr{C}_k^0(z_3)=\sum_{n_3\geqq 1}c_k\begin{pmatrix}n_3 & 0\\ 0 & 1\end{pmatrix}e^{2\pi i n_3 z_3} \tag{7.32}$$

and

$$\mathscr{C}_k^1(z_3)=\sum_{n_3\geqq 1}c_k\begin{pmatrix}n_3 & 1/2\\ 1/2 & 1\end{pmatrix}e^{2\pi i\left(n_3-\frac{1}{4}\right)z_3} \tag{7.33}$$

so that we have

$$\begin{pmatrix}(\chi_{10})_1(3)\\(\chi_{12})_1(3)\end{pmatrix}=e^{2\pi i z_2}\begin{pmatrix}\mathscr{C}^0_{10}(z_3)\mathscr{C}^1_{10}(z_3)\\\mathscr{C}^0_{12}(z_3)\mathscr{C}^1_{12}(z_3)\end{pmatrix}\begin{pmatrix}\Theta_0(z_1,z_3)\\\Theta_1(z_1,z_3)\end{pmatrix}. \tag{7.34}$$

To invert this system and obtain

$$\begin{pmatrix}\Theta_0(z_1,z_3)\\\Theta_1(z_1,z_3)\end{pmatrix}=e^{-2\pi i z_2}\begin{pmatrix}\mathscr{D}^0_{10}(z_3)\mathscr{D}^0_{12}(z_3)\\\mathscr{D}^1_{10}(z_3)\mathscr{D}^1_{12}(z_3)\end{pmatrix}\begin{pmatrix}(\chi_{10})_1(3)\\(\chi_{12})_1(3)\end{pmatrix} \tag{7.35}$$

we need only that $\mathscr{C}^0_{10}(z_3)\mathscr{C}^1_{12}(z_3)-\mathscr{C}^1_{10}(z_3)\mathscr{C}^0_{12}(z_3)$ be nonzero.

From the tables of Resnikoff and Saldaña [39] we have the following expansions:

$$\left.\begin{aligned}\mathscr{C}^0_{10}(z_3)&=\tfrac{1}{4}(2q-36q^2+272q^3-\dots)\\\mathscr{C}^1_{10}(z_3)&=\tfrac{1}{4}q^{-1/4}(-1q+16q^2-99q^3+\dots)\\\mathscr{C}^0_{12}(z_3)&=\tfrac{1}{12}(10q-132q^2+736q^3+\dots)\\\mathscr{C}^1_{12}(z_3)&=\tfrac{1}{12}q^{-1/4}(1q-88q^2+1275q^3+\dots).\end{aligned}\right\} \tag{7.36}$$

From these we find that

$$\mathscr{C}^0_{10}(z_3)\mathscr{C}^1_{12}(z_3)-\mathscr{C}^1_{10}(z_3)\mathscr{C}^0_{12}(z_3)=\tfrac{1}{4}q^{-1/4}(q^2-42q^3+819q^4+\dots), \tag{7.37}$$

which shows that Θ_0 and Θ_1 can be expressed as in (7.35). From (5.42), (5.43), (5.25), and (5.26) we also have

$$\chi^{\omega^0_1}(3)=e^{2\pi i z_2}q^{-1/4}\phi^{-1}(q)\Theta_1(z_1,z_3) \tag{7.38}$$

and

$$\chi^{\omega^0_2}(3)=e^{2\pi i z_2}\phi^{-1}(q)\Theta_0(z_1,z_3). \tag{7.39}$$

Combined with (7.35) this gives

$$\chi^{\omega^0_1}(3)=4q^{-2}(1+42q+945q^2+\dots)\phi^{-1}(q)[-\mathscr{C}^0_{12}(z_3)(\chi_{10})_1(3)+\mathscr{C}^0_{10}(z_3)(\chi_{12})_1(3)]$$

and (7.40)

$$\chi^{\omega^0_2}(3)=4q^{-7/4}(1+42q+945q^2+\dots)\phi^{-1}(q)[\mathscr{C}^1_{12}(z_3)(\chi_{10})_1(3)-\mathscr{C}^1_{10}(z_3)(\chi_{12})_1(3)].$$

Of course, $\phi^{-1}(q)=\sum_{n\geqq 0}p(n)q^n$, where p is the classical partition function, is a power series in q with lowest term 1. From this and (7.36) we see that we may write

$$\chi^{\omega^0_1}(3)=E^1_{10}(z_3)(\chi_{10})_1(3)+E^1_{12}(z_3)(\chi_{12})_1(3)$$

and (7.41)

$$\chi^{\omega^0_2}(3)=E^2_{10}(z_3)(\chi_{10})_1(3)+E^2_{12}(z_3)(\chi_{12})_1(3),$$

where

$$E^j_k(z_3)=q^{-1}\sum_{n\geqq 0}e^j_k(n)q^n \tag{7.42}$$

for $k=10,12$, $j=1,2$, with $e^j_k(n)\in\mathbb{Q}$ and $e^j_k(0)\neq 0$. This establishes the claim.

Furthermore, we may express

$$(\chi^{\omega_1^0})^j(\chi^{\omega_2^0})^{m-j}=q^{-m}\sum_{i=0}^{m}F_i(q)(\chi_{10})_1^{m-i}(\chi_{12})_1^i \tag{7.43}$$

for $0\leqq j\leqq m$, where $F_i(q)$ is a power series in q.

We wish to emphasize the significance of the fact that this series begins with q^{-m}. This is responsible for the isomorphism of $\mathfrak{M}_k^2(m)$ with $\mathcal{M}_k'(m)$.

We know that $(\chi^{\omega_1^0})^{n_1}$ and $(\chi^{\omega_2^0})^{n_2}$ are the characters of the $n_1{}^{\text{th}}$ and $n_2{}^{\text{th}}$ tensor powers of the standard $\mathfrak{F}_0^e$-modules with highest weights ω_1^0 and ω_2^0, respectively, and that $(\chi^{\omega_1^0})^{n_1}(\chi^{\omega_2^0})^{n_2}$ is the character of the tensor product of the two tensor powers just mentioned. This tensor product decomposes into the direct sum of standard modules [16] on level n_1+n_2. In the decomposition of this tensor product the $\mathfrak{F}_0^e$-module with highest weight $n_1\omega_1^0+n_2\omega_2^0$ is known to occur with multiplicity 1, and the only other $\mathfrak{F}_0^e$-modules which occur have highest weights of the form

$$n_1\omega_1^0+n_2\omega_2^0-i_1\alpha_1-i_2\alpha_2 \quad \text{for} \quad i_1, i_2\in\mathbb{Z}^+.$$

Thus, all weights which occur in the $\mathfrak{F}_0^e$-module with character $(\chi^{\omega_1^0})^{n_1}(\chi^{\omega_2^0})^{n_2}$ are of that form. This means that on level m, $\lambda=j\omega_1^0+(m-j)\omega_2^0$ occurs with multiplicity 1 in the module with character $(\chi^{\omega_1^0})^j(\chi^{\omega_2^0})^{m-j}$ but does not occur in any module with character $(\chi^{\omega_1^0})^k(\chi^{\omega_2^0})^{m-k}$ for $k<j$. Since

$$j\omega_1^0+(m-j)\omega_2^0=j\gamma_1^*+m\gamma_2^*$$

we may say that $e^{2\pi i(jz_1+mz_2)}$ occurs with coefficient 1 in the Fourier decomposition of $(\chi^{\omega_1^0})^j(\chi^{\omega_2^0})^{m-j}$ but does not occur in that of $(\chi^{\omega_1^0})^k(\chi^{\omega_2^0})^{m-k}$ for any $k<j$.

Therefore, any level m $\mathfrak{F}_0^e$-character $\chi(\mathfrak{z})$ can be uniquely expressed as a homogeneous degree m polynomial in $\chi^{\omega_1^0}$ and $\chi^{\omega_2^0}$ whose coefficients are power series in $q=e^{2\pi i z_3}$, namely,

$$\chi(\mathfrak{z})=\sum_{j=0}^{m}T_j(z_3)(\chi^{\omega_1^0})^j(\chi^{\omega_2^0})^{m-j} \tag{7.44}$$

for

$$T_j(z_3)=\sum_{n_3\geqq 0}t_j(n_3)e^{2\pi i n_3 z_3}. \tag{7.45}$$

The uniqueness of the decomposition (7.44) can also be shown using the fact that the fundamental characters are proportional to Jacobi theta functions [see (7.38), (7.39)].

Combining (7.44) with (7.43) we may write any level m $\mathfrak{F}_0^e$-character $\chi(\mathfrak{z})$ as

$$\chi(\mathfrak{z})=\sum_{j=0}^{m}H_j(q)(\chi_{10})_1^{m-j}(\chi_{12})_1^j, \tag{7.46}$$

where $H_j(q)=q^{-m}G_j(q)$ and $G_j(q)$ is a power series in q which coefficients in $\mathbb{C}$. We can also see easily that if $\chi(\mathfrak{z})\in\mathcal{M}_k'(m)$ then the power series $G_j(q)$ for each j has lowest nonzero term of degree at least m, so that $H_j(q)$ is a power series in q.

Consider the $(m+1)$-dimensional vector space V_m spanned by the functions $(\chi^{\omega_1^0})^j(\chi^{\omega_2^0})^{m-j}$, $0\leqq j\leqq m$, over the field $\mathcal{F}$ of Laurent series in q. The discussion leading up to (7.44) shows that these $m+1$ monomials are a basis for V_m, (7.46) shows that $\mathscr{M}_k(m) \subseteq V_m$, and (7.43) shows that the $(m+1)$ functions $(\chi_{10})_1^j(\chi_{12})_1^{m-j}$, $0 \leqq j \leqq m$, also span V_m over $\mathscr{F}$, so that they are also a basis for V_m over $\mathscr{F}$. Thus the expression (7.46) is uniquely determined for each $\chi(\mathfrak{z}) \in \mathscr{M}_k(m)$.

For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ we have $\bar{g} = \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ \hline c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \in Sp_2(\mathbb{Z})$.

Replacing $\mathfrak{z}$ by $\bar{g}\cdot\mathfrak{z}$ and z_3 by $g \cdot z_3$ in (7.46) we get

$$\chi(\bar{g}\cdot\mathfrak{z}) = \sum_{j=0}^{m} H_j(g\cdot z_3)(\chi_{10})_1^{m-j}(\bar{g}\cdot\mathfrak{z})(\chi_{12})_1^j(\bar{g}\cdot\mathfrak{z}). \tag{7.47}$$

Thus, we get

$$\chi(\mathfrak{z}) = \sum_{j=0}^{m} H_j(g\cdot z_3)(cz_3+d)^{10(m-j)+12j-k}(\chi_{10})_1^{m-j}(\mathfrak{z})(\chi_{12})_1^j(\mathfrak{z}), \tag{7.48}$$

where we have written $H_j(q)$ as $H_j(z_3)$. Comparing this with (7.46) we see that for each j, $0 \leqq j \leqq m$,

$$H_j(z_3) = H_j(g\cdot z_3)(cz_3+d)^{10(m-j)+12j-k}. \tag{7.49}$$

That is, the Laurent series $H_j(z_3) = e^{-2\pi i m z_3} G_j(z_3)$ occurring as coefficients of the monomials $(\chi_{10})_1^{m-j}(\chi_{12})_1^j$ in the expansion of $\chi(\mathfrak{z}) \in \mathscr{M}_k(m)$ transform as modular functions of genus 1 with weight $k-(10(m-j)+12j)$. In the case when $\chi(\mathfrak{z}) \in \mathscr{M}_k'(m)$, $H_j(z_3)$ is a power series in $e^{2\pi i z_3}$, so is holomorphic at ∞ and is actually a genus 1 modular form of weight $k-(10(m-j)+12j)$. From the classical result (6.10), we may thus express each $H_j(z_3)$ as a polynomial in $\phi_4^a \phi_6^b$ with

$$4a+6b = k-(10(m-j)+12j),$$

and then we have that

$$\chi(\mathfrak{z}) = \sum_{c+d=m} \sum_{4a+6b+10c+12d=k} \alpha_{abcd} \phi_4^a \phi_6^b (\chi_{10})_1^c (\chi_{12})_1^d \tag{7.50}$$

for some $\alpha_{abcd} \in \mathbb{C}$. Thus,

$$L_m'\left(\sum_{c+d=m} \sum_{4a+6b+10c+12d=k} \alpha_{abcd} \Psi_4^a \Psi_6^b \chi_{10}^c \chi_{12}^d\right) = \chi(\mathfrak{z}) \tag{7.51}$$

establishes the surjectivity of L_m' onto $\mathscr{M}_k'(m)$. Injectivity was established when the monomials $(\chi_{10})_1^j(\chi_{12})_1^{m-j}$ were seen to be a basis of V_m over the field of Laurent series $\mathscr{F}$.

Corollary 7.10. *For each* $m \geqq 0$

$$\dim(\mathscr{M}_k'(m)) = \dim(\mathfrak{M}_k^2(m)) = \text{Cardinality of the set } S(k,m)$$

given in (7.22).

Corollary 7.11. *The space* $\mathscr{M}' = \sum_{k \geqq 0} \sum_{m \geqq 0} \mathscr{M}_k'(m)$ *has the structure of a graded ring and is isomorphic to the Igusa ring* $C[\Psi_4, \Psi_6, \chi_{10}, \chi_{12}]$.

Theorem 7.9 shows what a special role is played by the space $\mathcal{M}_k'$. In Proposition 5.3 we saw a natural correspondence between the characters of $\mathfrak{F}$-dominant $\mathfrak{F}_0^e$-modules in category $\mathscr{C}_0'$ and $\mathfrak{F}$-modules in $\mathscr{C}$. Parallel to that we have from Theorem 7.9 a correspondence between the space $\mathcal{M}_k'$ of weight k $\mathrm{PSL}_2(\mathbb{Z})$-invariant $\mathfrak{F}$-dominant $\mathfrak{F}_0^e$-characters and the space of $\mathfrak{F}$-characters which are weight k Siegel modular forms, namely, $\mathfrak{M}_k^2$, whose dimension is given in (1.25).

The "lifting" of elliptic modular forms to Siegel modular forms has attracted the attention of several authors [2, 20, 22, 38, 43]. Our work has been particularly inspired by the work of Maass.

In a series of papers [30–32] Maass studied the subspace $\mathfrak{L}_k^2 \subseteqq \mathfrak{M}_k^2$ of forms

$$f(\mathfrak{Z}) = \sum_{0 \leqq N \in S_2^*(\mathbb{Z})} A(N) e^{2\pi i \operatorname{Tr}(N\mathfrak{Z})} \tag{7.52}$$

satisfying the special coefficient relation

$$A\begin{pmatrix} n_3 & n_1/2 \\ n_1/2 & n_2 \end{pmatrix} = \sum_{\substack{d \mid (n_1, n_2, n_3) \\ d>0}} d^{k-1} A \begin{pmatrix} \dfrac{n_2 n_3}{d^2} & \dfrac{n_1}{2d} \\ \dfrac{n_1}{2d} & 1 \end{pmatrix}. \tag{7.53}$$

Such a relation had been observed in the extensive coefficient tables of Resnikoff and Saldaña [39] for some Eisenstein series Ψ_k and was proven in [28] by Maass for all Eisenstein series. Maass then considered the subspace $\mathfrak{S}_k^2$ of parabolic forms in $\mathfrak{L}_k^2 = \mathbb{C}\Psi_k + \mathfrak{S}_k^2$. He computed the dimension of these spaces and showed that the relation (7.53) is equivalent to a simple construction of $f(\mathfrak{Z})$ out of Hecke operators applied to the level 1 of $f(\mathfrak{Z})$, namely $f_1(\mathfrak{Z}) \in \mathcal{M}_k(1)$. Maass found the dimension of $\mathfrak{S}_k^2$ to be equal to the cardinality of the set $S(k, 1)$, and the dimension of $\mathfrak{L}_k^2$ to be one greater than that. The construction using Hecke operators of forms in $\mathfrak{L}_k^2$ shows that any given $f_1(\mathfrak{Z}) \in \mathcal{M}_k(1)$ determines a unique $f(\mathfrak{Z}) \in \mathfrak{L}_k^2$ whose level 1 slice is $f_1(\mathfrak{Z})$. Let $f_1(\mathfrak{Z})$ be written as

$$e^{2\pi i z_2} \sum_{\substack{4n_3 \geqq n_1^2 \\ n_3 \geqq 0}} A(n_1, 1, n_3) e^{2\pi i (n_1 z_1 + n_3 z_3)}.$$

Then the corresponding $f(\mathfrak{Z})$ is in $\mathfrak{S}_k^2$ if and only if $A(0,1,0)=0$. So on level 1 the subspace of $\mathcal{M}_k(1)$ consisting of those $f_1(\mathfrak{Z})$ such that $A(0,1,0)=0$ has the same dimension as the space $\mathfrak{M}_k^2(1) \approx \mathcal{M}_k'(1)$.

The work of Maass is related to our Theorem 7.9 in the case when $m=1$. In the case when $m>1$ it is not known how to use Hecke operators to construct from a given $f_m(\mathfrak{Z}) \in \mathcal{M}_k'(m)$ a form $f(\mathfrak{Z}) \in \mathfrak{M}_k^2$ such that $L_m(f(\mathfrak{Z})) = f_m(\mathfrak{Z})$. Our theory indicates that such a construction may exist.

In order to relate our work to that of Maass we would like to show his level 1 construction using the Hecke operators $T_k(m)$ defined in (6.18). Recall that the genus 1 Eisenstein series of weight k is

$$\varphi_k(z_3) = 1 + \frac{(2\pi i)^k}{(k-1)!\,\zeta(k)} \sum_{n_3 \geqq 1} \sigma_{k-1}(n_3) e^{2\pi i n_3 z_3}, \tag{7.54}$$

where ζ is the Riemann zeta function and $\sigma_{k-1}(n_3) = \sum_{d \mid n_3} d^{k-1}$ is the divisor power function.

Theorem 7.12 (Maass). *Let*

$$f_1(z_1,z_2,z_3)=e^{2\pi i z_2}\sum_{\substack{4n_3\geqq n_1^2\\ n_3\geqq 0}} A(n_1,1,n_3)e^{2\pi i(n_1z_1+n_3z_3)}$$

be a function on level 1 *satisfying condition (b) of Proposition 7.5, and let*

$$f(z_1,z_2,z_3)=A(0,1,0)\frac{(k-1)!\,\zeta(k)}{(2\pi i)^k}\varphi_k(z_3)+\sum_{m\geqq 1}(T_k(m)f_1)(z_1,z_2,z_3).$$

Then $f_1(z_1,z_2,z_3)\in\mathscr{M}_k(1)$ *if and only if* $f(z_1,z_2,z_3)\in\mathfrak{L}_k^2$. *Also,* $f_1(z_1,z_2,z_3)\in\mathscr{M}_k'(1)$ *if and only if* $f(z_1,z_2,z_3)\in\mathfrak{S}_k^2$.

Proof. As shown at the end of Sect. 6, $T_k(m):\mathscr{M}_k(1)\to\mathscr{M}_k(m)$, so if $f_1\in\mathscr{M}_k(1)$ we have that for $m\geqq 1$, $L_m(f)=T_k(m)f_1\in\mathscr{M}_k(m)$ satisfies Proposition 7.5(a) and (b), as does

$$L_0(f)=A(0,1,0)\frac{(k-1)!\,\zeta(k)}{(2\pi i)^k}\varphi_k(z_3)\in\mathscr{M}_k(0)=\mathfrak{M}_k^1.$$

So $f(z_1,z_2,z_3)$ satisfies (a) and (b). Let us write

$$f_1(z_1,z_2,z_3)=e^{2\pi i z_2}\sum_{n_3\geqq 0}\sum_{n_1\in\mathbb{Z}} A(n_1,1,n_3)e^{2\pi i(n_1z_1+n_3z_3)}. \tag{7.55}$$

Then we have

$$\begin{aligned}(T_k(m)f_1)&(z_1,z_2,z_3)\\ &=e^{2\pi i m z_2}\sum_{n_3\geqq 0}\sum_{n_1\in\mathbb{Z}}\sum_{\substack{d\mid m\\ d\geqq 1}}\sum_{0\leqq b<d} m^{k-1}d^{-k}A(n_1,1,n_3)e^{2\pi i\left(n_1\frac{m}{d}z_1+n_3\frac{m}{d^2}z_3+\frac{n_3b}{d}\right)}\\ &=e^{2\pi i m z_2}\sum_{n_3\geqq 0}\sum_{n_1\in\mathbb{Z}}\sum_{\substack{d\mid m,n_3\\ d\geqq 1}}\left(\frac{m}{d}\right)^{k-1}A(n_1,1,n_3)e^{2\pi i\left(n_1\frac{m}{d}z_1+n_3\frac{m}{d^2}z_3\right)}.\end{aligned} \tag{7.56}$$

We have used the fact that

$$\sum_{0\leqq b<d} e^{2\pi i\frac{n_3b}{d}}=\begin{cases} d & \text{if } d\mid n_3\\ 0 & \text{if } d\nmid n_3.\end{cases} \tag{7.57}$$

Putting $ad=m$ and replacing n_3 by dn_3 in (7.56) we have

$$(T_k(m)f_1)(\mathfrak{Z})=e^{2\pi i z_2}\sum_{n_3\geqq 0}\sum_{n_1\in\mathbb{Z}}\sum_{\substack{a\mid m\\ a\geqq 1}} a^{k-1}A\left(n_1,1,\frac{mn_3}{a}\right)e^{2\pi i(n_1az_1+n_3az_3)}. \tag{7.58}$$

Then

$$\begin{aligned}f(\mathfrak{Z})&=\sum_{m\geqq 1}(T_k(m)f_1)(\mathfrak{Z})\\ &=\sum_{m\geqq 1}e^{2\pi i m z_2}\sum_{n_3\geqq 0}\sum_{n_1\in\mathbb{Z}}\sum_{\substack{a\mid m\\ a\geqq 1}} a^{k-1}A\left(n_1,1,\frac{mn_3}{a}\right)e^{2\pi i(n_1az_1+n_3az_3)}.\end{aligned} \tag{7.59}$$

If we set this equal to the Fourier series $\sum_{0\leqq N\in S_2'(\mathbb{Z})} A(N)e^{2\pi i(m_1z_1+m_2z_2+m_3z_3)}$ for $N=\begin{pmatrix} m_3 & m_1/2\\ m_1/2 & m_2\end{pmatrix}$ and compare coefficients then we get the Maass coefficient relation (7.53). This relation shows that for $n_2>0$, $A(n_1,n_2,n_3)=A(n_1,n_3,n_2)$, and that $A(0,0,n_3)=A(0,1,0)\sigma_{k-1}(n_3)=A(0,n_3,0)$, so $f(\mathfrak{Z})$ satisfies condition (c), symmetry in z_2 and z_3, and by Proposition 7.5 we have $f(\mathfrak{Z})\in\mathfrak{M}_k^2$. Since $f(\mathfrak{Z})$ satisfies the Maass condition (7.53), $f(\mathfrak{Z})\in\mathfrak{L}_k^2$. The converse is clear, as is the fact that $f(\mathfrak{Z})$ is parabolic if and only if $A(0,1,0)=0$, that is, $f_1(\mathfrak{Z})\in\mathcal{M}_k'(1)$.

Theorem 7.13 (Maass). *We have for* $k\geqq 4$ *even*, $\dim(\mathcal{M}_k(1))=\dim(\mathfrak{L}_k^2)=\left[\frac{k+2}{6}\right]$ *and* $\dim(\mathcal{M}_k'(1))=\dim(\mathfrak{S}_k^2)=\left[\frac{k-4}{6}\right]$ *which equals the cardinality of the set* $S(k,1)$ *given in* (7.22).

Remark. Maass established these results in [30, 31] based on the computation of the dimension of the vectorial modular forms $\begin{pmatrix}c_0(z_3)\\c_1(z_3)\end{pmatrix}$ satisfying (5.45) which determine an element of $\mathcal{M}_k(1)$. The condition (5.45) actually shows that $c_0(z_3)$ determines $c_1(z_3)$. Maass, however, used a different notation than ours. He used the classical Jacobi theta functions given in (5.40) to express a level 1 function rather than the fundamental $\mathfrak{F}_0^e$-characters $\bar{\chi}^{\omega_1^0}$ and $\bar{\chi}^{\omega_2^0}$ as we have done. One result of this difference is that Maass put the factor $\eta^{-1}(e^{2\pi i z_3})$ into the coefficients c_0 and c_1 which made his vectorial modular form $\begin{pmatrix}c_0\\c_1\end{pmatrix}$ have weight $k-\frac{1}{2}$. For us that form has weight k.

The computation of $\dim(\mathfrak{S}_k^2)$ in [31] is based on results of Petersson in [37]. Corollary 7.10 gives a dimension formula for $\mathcal{M}_k'(m)$ for all $m\geqq 1$ which generalizes Maass' result and allows each $f(\mathfrak{Z})\in\mathfrak{M}_k^2$ to be a "lifing" of a vectorial modular form $\begin{pmatrix}c_0(z_3)\\c_1(z_3)\\\vdots\\c_M(z_3)\end{pmatrix}$ of weight k satisfying the conditions of Proposition 5.4. We may show that each $c_j(z_3)$ is an automorphic form of weight k with multiplier system with respect to a subgroup of the congruence subgroup $\Gamma_0(4(M+2))$. Recalling the notation of (5.24)–(5.26) and Proposition 5.4 we see that if we write a level M function as

$$f_M(\mathfrak{Z})=\sum_{j=0}^{M}\bar{\chi}^{j\omega_1^0+(M-j)\omega_2^0}(\mathfrak{Z})c_j(z_3) \tag{7.60}$$

then $f_M(\mathfrak{Z})\in\mathcal{M}_k(M)$ if and only if the vectorial function $\mathfrak{C}(z_3)=\begin{pmatrix}c_0(z_3)\\c_1(z_3)\\\vdots\\c_M(z_3)\end{pmatrix}$ satisfies the conditions

$$\mathfrak{C}\left(\frac{-1}{z_3}\right)(-z_3)^{-k}=U\mathfrak{C}(z_3) \tag{7.61}$$

and

$$\mathfrak{C}(z_3+1)=V\mathfrak{C}(z_3), \tag{7.62}$$

where $U=(U_{ij})$ and $V=(V_{ij})$ are $(M+1)\times(M+1)$ matrices such that for $1\leqq i\leqq M+1$ and $1\leqq j\leqq M+1$

$$U_{ij}=\sqrt{\frac{2}{M+2}}\sin\left(\frac{\pi ij}{M+2}\right) \tag{7.63}$$

and

$$V_{jj}=e^{-\left(\frac{j^2}{4(M+2)}-\frac{1}{8}\right)2\pi i}, \quad V_{ij}=0 \quad \text{for} \quad i\neq j. \tag{7.64}$$

This is equivalent to saying that there is a group homomorphism τ from $\mathrm{PSL}_2(\mathbb{Z})$ to $\mathrm{GL}_{M+1}(\mathbb{C})$ such that

$$\mathfrak{C}\left(\frac{az_3+b}{cz_3+d}\right)(cz_3+d)^{-k}=\tau\begin{pmatrix}a & b\\ c & d\end{pmatrix}\mathfrak{C}(z_3) \tag{7.65}$$

for all $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{PSL}_2(\mathbb{Z})$ such that

$$\tau\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=U \quad \text{and} \quad \tau\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=V. \tag{7.66}$$

The subgroup of $\mathrm{PGL}_2(\mathbb{Z})$ consisting of all elements g such that $\tau(g)$ is diagonal is the subgroup under which each function $c_j(z_3)$ will be automorphic of weight k with a certain multiplier system.

We can certainly say that $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ is in that subgroup. Also, from (7.64) we see that $V_{jj}^{4(M+2)}=(-1)^{M+2}$. So $\tau\begin{pmatrix}1 & 4(M+2)\\ 0 & 1\end{pmatrix}=\pm I$ and therefore,

$$\tau\begin{pmatrix}1 & 0\\ 4(M+2) & 1\end{pmatrix}=\tau\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & -4(M+2)\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\right)=U(\pm I)U^{-1}=\pm I$$

which is diagonal. A deeper investigation of this subgroup is carried out by Kac and Peterson in [18].

8. Future Prospects

We would like to indicate briefly in this last section some directions for further research which we hope to pursue in subsequent publications.

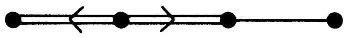
Within algebra $\mathfrak{F}$ there are other subalgebras besides $\mathfrak{F}_0$ with respect to which one may decompose $\mathfrak{F}$ and $\mathfrak{F}$-characters. We have in mind some rank 2 hyperbolic Kac-Moody algebras found by Lepowsky and Moody in [24] to be connected to Hilbert modular surfaces. We expect that such decompositions give a Lie-theoretical explanation of the connection between Siegel modular forms of genus 2 and Hilbert modular forms associated with quadratic number fields (see [39]). Geometrically one sees that slicing the Siegel domain (cone) by $\mathfrak{F}_0$ gives parabolas while slicing it by hyperbolic rank 2 subalgebras should give hyperbolas.

We may generalize the construction given in Sect. 4 of a hyperbolic algebra $\bar{\mathfrak{G}}=\mathfrak{G}(\mathfrak{g}^{\wedge e}, V)$ from an affine algebra $\mathfrak{g}^{\wedge}$ and its basic module V. In the case when $\mathfrak{g}^{\wedge}$

is of ADE type the basic module V has a nice construction (see [10, 17]) and action of $\hat{\mathfrak{g}}$ on V is given by vertex operators. One may expect that a result such as Theorem 3.2 can be established in that generality which would shed light on root multiplicities of $\mathfrak{G}$.

The interesting Kac-Moody algebras will be those whose Weyl groups are related to classically studied arithmetic groups. The rank 3 hyperbolic algebras have already been seen by ourselves and Yoshida [42] to have hyperbolic triangle groups for Weyl groups. The representation theory of these algebras should shed light on the theory of modular forms with respect to certain arithmetic subgroups of $Sp_2(\mathbb{R})$.

Our construction of $\mathfrak{G}$ when applied to $\hat{\mathfrak{g}}$ of type $C_2^{(1)}$ gives an algebra with Dynkin diagram shown in (1.27), and when applied to $\hat{\mathfrak{g}}$ of type $D_3^{(2)}$ gives an algebra with diagram


. (8.1)

Both algebras have Weyl group W with generators A, B, C, D having relations

$$\begin{aligned} &1 = A^2 = B^2 = C^2 = D^2 , \\ &1 = (AB)^4 = (AC)^2 = (AD)^2 = (BC)^4 = (BD)^2 = (CD)^3 . \end{aligned} \tag{8.2}$$

These are the relations among the four generators of the Klein-Fricke group Ψ_1^* which contains the Picard group $\Psi_1 = PSL_2(\mathbb{Z}[i])$ as a subgroup of index 4. (See Magnus [34, p. 152], but there are errors in his list of relations.) Ψ_1^* is generated by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ where

$$\begin{aligned} &\mathcal{A}: z \to \bar{z} \\ &\mathcal{B}: z \to -i\bar{z} \\ &\mathcal{C}: z \to -\bar{z} - 1 \\ &\mathcal{D}: z \to 1/\bar{z} . \end{aligned} \tag{8.3}$$

One or both of these algebras should be closely related to the theory of Hermitian modular forms of genus 2. In particular, the lifting of $\hat{\mathfrak{g}}^e$-characters which are $PSL_2(\mathbb{Z}[i])$-invariant to $\mathfrak{G}$-characters which are $Sp_2(\mathbb{Z}[i])$-invariant should be similar to the lifting discussed in this paper. Already Kojima [21] has investigated this lifting for level 1 using Hecke operators as suggested in the work of Maass [30–32]. We believe that the existence of lifting and descent operations between modular forms of various types should be explained by the presence of a Kac-Moody algebra and a subalgebra.

Finally, we should mention that there are many questions concerning just the algebra $\mathfrak{F}$ which remain unanswered. One would like to have complete information about the root multiplicities and a deeper understanding of the structure of $\mathfrak{F}$.

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Note added in proof. We have recently learned through the referee of the paper "On the theory of Jacobi forms" by M. Eichler and D. Zagier. The authors, inspired by the work of Maass [29–32], undertook the construction of a new theory. Jacobi forms (of even weight) in our language are certain characters of affine Lie algebra $\mathfrak{F}_0$. This fact naturally entailed some overlap with the second part of our paper (Sects. 5–7). These two different points of view can be very fruitful for further results and generalizations. We are grateful to the referee for informing us about the work of Eichler and Zagier, and to Zagier for sending us their preprint before publication.

We have also received a preprint "Kac-Moody symmetry of gravitation and supergravity theories" by B. Julia (l'Ecole Normale Supérieure) in which he conjectures that the hyperbolic algebra $\mathfrak{F}$ may be an algebra of internal symmetries of Einstein gravitational equations. It has been known for about ten years that the subalgebra $\mathfrak{F}_0$ is responsible for the internal symmetries of gravitational plane waves. Thus, the highly abstract Lie algebra $\mathfrak{F}$ promises to describe fundamental symmetries of nature.