FUSION ALGEBRAS, SYMMETRIC POLYNOMIALS, ORBITS OF N-GROUPS, AND RANK-LEVEL DUALITY

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Abstract

A method of computing fusion coefficients for Lie algebras of type A_{n-1} on level k was recently developed by A. Feingold and M. Weiner [FW] using orbits of $\mathbb{Z}_n^k$ under the permutation action of S_k on k-tuples. They got the fusion coefficients only for n = 2 and 3. We will extend this method to all $n \geq 2$ and all $k \geq 1$. First we show a connection between Young diagrams and S_k-orbits of $\mathbb{Z}_n^k$, and using Pieri rules we prove that this method works for certain specific weights that generate the fusion algebra. Then we show that the orbit method does not work in general, but with the help of the Jacobi-Trudi determinant, we give an iterative method to reproduce all type A fusion products.

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Contents

Chapter 1

Introduction

Fusion algebras play a very important role in conformal field theory [Fu1] and in the theory of vertex operator algebras [FZ]. Many equivalent interpretations have been found in other mathematical contexts such as quantum groups, Hecke algebras at roots of unity [GW], quantum cohomology of the Grassmanian [BCF] and quite a few other areas.

There is a finite dimensional fusion algebra associated with an affine Kac-Moody algebra and a level $1 \leq k \in \mathbb{Z}$, with distinguished basis indexed by the level k weights. The structure constants $N_{a,b}^{(k)c}$ for such an algebra, also known as the fusion coefficients, can be expressed by the Kac-Walton algorithm ([Kac] and [Wa1]) as an alternating sum of tensor product coefficients, or as a modification of the Racah-Speiser algorithm for tensor products.

This algorithm does not give any insight into why the structure constants for the fusion algebra are non-negative. This fact comes from other theories such as vertex operator algebras where the fusion coefficients are the dimension of spaces of intertwining operators.

Many authors have tried to find a closed formula to express the fusion coefficients, but despite all the work of mathematicians and physicists during the 1990's the results in this direction are few, and closed formulas are known only for small ranks. Gepner and Witten [GeWi] proved that for A_1

$$N_{a,b}^{(k)c} = \begin{cases} 1, & \text{if } c = a+b \mod 2 \text{ and } |a-b| \leq c \leq \min\{a+b, 2k-a-b\} \\ 0, & \text{otherwise.} \end{cases}$$

For A_2 and A_3 closed formulas were given by Begin, Kirillov, Mathieu and Walton [BKMW] and [BMW]. They used combinatorial objects called Berenstein-Zelevinsky triangles. Another attempt at finding a closed formula was done by Feingold and Weiner [FW]. They used S_k-orbits of $\mathbb{Z}_2^k$ and $\mathbb{Z}_3^k$. For $a \in \mathbb{Z}_N^k$, let $[a]$ denote the S_k-orbit of a. There is a bijection between the level k weights λ of a type A_{N-1} affine Kac-Moody algebra and the S_k-orbits $[\lambda]$ of $\mathbb{Z}_N^k$. For A_1 and $k \geq 1$ they proved

$$N_{\mu,\lambda}^{(k)\nu} = M_{[\mu],[\lambda]}^{(k)[\nu]},$$

and for A_2 and $k \geq 1$ they proved

$$M_{[\mu],[\lambda]}^{(k)[\nu]} = \binom{N_{\mu,\lambda}^{(k)\nu} + 1}{2},$$

where $M_{[\mu],[\lambda]}^{(k)[\nu]}$ is the number of orbits of $T([\mu],[\lambda],[\nu]) = \{(x,y,z) \in [\mu] \times [\lambda] \times [\nu] \mid x + y = z\}$ under the action of S_k on $T([\mu],[\lambda],[\nu])$ given by S_k simultaneously acting on each k-tuple.

This work is an extension of the method of Feingold and Weiner to all ranks (A_N for $N \geq 3$) and all levels. Although this method does not give a general closed formula for high ranks (A_N for $N \geq 3$), it brings some progress to the theory of type A fusion algebras. Using S_k-orbits of $\mathbb{Z}_N^k$, the Feingold-Weiner algorithm gives a closed formula for fusion coefficients $N_{\lambda,\mu}^{(k)\nu} = M_{[\lambda],[\mu]}^{(k)[\nu]}$ where μ is a weight of the form (5.6). Weights of this form include a set of generators for the fusion algebra, so we can compute the rest of the structure constants iteratively by using the Jacobi-Trudi determinant.

The structure of this thesis is as follows. Chapter 2 gives the definition of an abstract fusion algebra due to Fuchs [Fu1], as well as preliminary material that leads to this definition. We also present a brief description of the Racah-Speiser and Kac-Walton algorithms for tensor and fusion products and we finish the chapter by setting up the notation we will follow for the rest of the thesis. In Chapter 3 we describe the method of Feingold and Weiner [FW] and give an interesting characterization for orbits with only zeroes and ones. In Chapter 4 we give a full description of the algebra of symmetric polynomials and its relation to the type A fusion algebras. We also describe the work of Goodman and Wenzl [GW] that shows that the fusion algebra of type A is isomorphic to a quotient of the algebra of symmetric polynomials . We also include a proof of the type A rank-level duality that says that $\mathcal{F}'(A_{N-1}, k) \cong \mathcal{F}'(A_{k-1}, N)$ where $\mathcal{F}'(A_{N-1}, k)$ is the quotient of the fusion algebra $\mathcal{F}(A_{N-1}, k)$ by the ideal that identifies the set of simple currents with the identity. We finish the chapter with some generalities about simple currents. Chapter 5 exhibits the connection between arithmetic of S_k-orbits of $\mathbb{Z}_N^k$ and fusion Pieri rules for symmetric polynomials and shows how the orbits method yields an algorithm for computing fusion coefficients for special weights. In Chapter 6 we show an application of the orbits method to tensor products by showing that classical Pieri rules are equivalent to the orbit method for orbits with only zeroes and ones, a result that is an extension of the one for fusion products. We finish the thesis by exhibiting a different version of each the Racah-Speiser and Kac-Walton algorithms from the Young tableaux point view which are more practical than the classical algorithms for making computations by hand since, even for $N \geq 3$, computations can be done on a piece of paper.

Chapter 2

Preliminaries

In this chapter we give the definition of Lie algebras, the construction and classification of Kac-Moody Lie algebras, the definition of twisted and untwisted affine Lie algebras and its irreducible modules. We also set up the notation that we will follow for the rest of this work.

2.1 Kac-Moody Lie algebras

We begin by giving some important definitions that we will be using throughout the chapter.

Definition 2.1.1. *a) A* ***Lie algebra*** *is a vector space L over a field K (usually $K = \mathbb{R}$ or $\mathbb{C}$) equipped with a bilinear function $[\cdot,\cdot] : L \times L \to L$, called bracket, satisfying the conditions*

1) $[x, x] = 0, \quad$ *for all* $x \in L$.

2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \quad$ *for all* $x, y, z \in L$.

b) A subspace I of a Lie algebra L is called an ***ideal*** *if for all $x \in L$ and for all $y \in I$ we have that $[x, y] \in I$.*

c) We say that a Lie algebra L is ***simple*** *if the only ideals of L are 0 and L and*

$\dim(L) > 1$.

d) The center $\mathcal{C}$ *of a Lie algebra* L *is the ideal*

$$\mathcal{C} = \{y \in L \mid [x, y] = 0 \textit{ for all } x \in L\}.$$

e) We say that a vector space V *is module for a Lie algebra* L*, if there is a a bilinear function* $\cdot : L \times V \to V$ *satisfying*

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v),$$

for all $x, y \in L$ *and for all* $v \in V$*. If* V *is a module for a Lie algebra* L*, we say that* V *is an* L*-module.*

Example 2.1.2. *Consider the vector space* sl_N *of* $N \times N$ *matrices of trace 0 with the bracket defined by* $[A, B] = AB - BA$*. Under this bracket* sl_N *becomes a Lie algebra, and it can be proved that this algebra is a simple Lie algebra. The vector space* $V = K^N$ *is a module for* sl_N *under the action* $x \cdot v = xv$ *of matrix multiplication for* $x \in sl_N$ *and* $v \in K^N$*. We will see a construction of this Lie algebra in a later example in this chapter.*

We need the following definition before we present the definition of a Kac-Moody algebra.

Definition 2.1.3. *Given a matrix* $A = (a_{ij})_{i,j=1}^t$ *of rank* l*, we define a* ***realization*** *of* A *to be a triple* $(\mathfrak{h}, \Pi, \check{\Pi})$ *where* $\mathfrak{h}$ *is a complex vector space of dimension* $2t - l$*,* $\Pi = \{\alpha_1, ..., \alpha_t\} \subseteq \mathfrak{h}^*$ *and* $\check{\Pi} = \{\check{\alpha}_1, ..., \check{\alpha}_t\} \subseteq \mathfrak{h}$ *satisfying*

1) Π *and* $\check{\Pi}$ *are linearly independent,*

2) $\alpha_j(\check{\alpha}_i) = a_{ij}$*.*

We define a Lie algebra $\tilde{\mathfrak{g}}(A)$ with generators $e_i, f_i \quad (i = 1, ..., t)$ and $\mathfrak{h}$ satisfying the following relations

$$\begin{aligned} &[h, h'] = 0, \qquad \text{for all } h, h' \in \mathfrak{h}; \\ &[e_i, f_j] = \delta_{ij}\check{\alpha}_i, \qquad \text{for } i, j = 1, ..., t; \\ &[h, e_i] = \alpha_i(h)e_i, \qquad [h, f_i] = -\alpha_i(h)f_i, \quad \text{for all } h \in \mathfrak{h} \text{ and } i = 1, ..., t. \end{aligned} \tag{2.1}$$

The elements $e_i, f_i \quad (i = 1, ..., t)$ and the set $\mathfrak{h}$ are called the **Chevalley generators** of $\tilde{\mathfrak{g}}(A)$.

It can be proved that the Lie algebra $\tilde{\mathfrak{g}}(A)$ has a unique maximal ideal τ intersecting $\mathfrak{h}$ trivially. We can now define the Kac-Moody algebra associated with A.

Definition 2.1.4. *Given a matrix A, we define the associated* ***Kac-Moody algebra*** *as the Lie algebra $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)/\tau$ where τ is the unique maximal ideal intersecting $\mathfrak{h}$ trivially.*

Example 2.1.5. *Consider the $(N-1) \times (N-1)$ matrix*

$$A = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

Let $E_{i,j}$ be the $N \times N$ matrix with 1 the (i,j)-entry and 0 everywhere else and set

$$\check{\alpha}_i = E_{i,i} - E_{i+1,i+1} \quad i = 1, ..., N-1.$$

Let $\mathfrak{h} = Span\{\check{\alpha}_1, ..., \check{\alpha}_{N-1}\}$.

Define the linear functional ϵ_i by

$$\epsilon_i(diag(a_1, ..., a_N)) = a_i, \quad i = 1, ..., N,$$

and set

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad i = 1, ..., N-1,$$

then $\mathfrak{h}^* = Span\{\alpha_1, ..., \alpha_{N-1}\}$. Now, set $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$ for $i = 1, ..., N-1$. It can be verified that e_i, f_i and $\mathfrak{h}$ satisfy the relations (2.1)*. It can also be proved that the algebra $\tilde{\mathfrak{g}}(A)$ is simple, so in particular it has no non-zero ideals which intersect $\mathfrak{h}$ trivially. Therefore $\mathfrak{g}(A) = \tilde{\mathfrak{g}}(A)$.*

This Lie algebra is isomorphic to the Lie algebra of traceless $N \times N$ matrices defined in the previous example.

Next we give the definition of generalized Cartan matrix.

The classification of Kac-Moody algebras is known for a special type of matrices known as generalized Cartan matrices, which we now define.

Definition 2.1.6. *An integral $N \times N$ matrix $A = (a_{ij})_{i,j=1}^N$ is called a* ***generalized Cartan matrix*** *if satisfies the following conditions:*

1) $a_{ii} = 2$ for $1 \leq i \leq N$,

2) $a_{ij} \leq 0$ for $i \neq j$,

3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Before stating the classification of Kac-Moody algebras associated to generalized Cartan matrices, we need the following definition.

Definition 2.1.7. *A matrix is called* ***decomposable*** *if, after some reordering of the indexes, the matrix has the form*

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

If a matrix is not decomposable will be called ***indecomposable***.

Notation 2.1.8. *Let $u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \in \mathbb{R}^N$. We write $u > 0$, if $u_i > 0$ for $i = 1, ..., N$ and $u < 0$ if $u_i < 0$ for $i = 1, ..., N$.*

There are finite and infinite dimensional Kac-Moody Lie algebras. Every finite dimensional simple Lie algebra over $\mathbb{C}$ is a Kac-Moody algebra, and their classification, developed at the begining of the previous century, is well known. The infinite dimensional Kac-Moody algebras are classified into two types, affine and indefinite. The affine are divided into twisted and untwisted and are well understood. There is less known about the indefinite type. Given a matrix A, one can tell if the Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ is finite, affine or indefinite. The classification is due to Vinberg [Vin]. (See also [Wan] and [Kac].)

Theorem 2.1.9. *[Vin] Let A be an indecomposable generalized Cartan matrix, and $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac-Moody algebra built from A. Then $\mathfrak{g}$ is one of the following three types.*

a) $\mathfrak{g}$ is finite if and only if $\det A \neq 0$; $Au > 0$ for some $u > 0$; $Au \geq 0$ implies $u = 0$ or $u > 0$.

b) $\mathfrak{g}$ is affine if and only if corank$(A) = 1$; $Au = 0$ for some $u > 0$; $Au \geq 0$ implies $Au = 0$.

c) $\mathfrak{g}$ is indefinite if and only if $Au < 0$ for some $u > 0$; $Au \geq 0$ and $u \geq 0$ implies $u = 0$.

Definition 2.1.10. *Let A be an indecomposable generalized Cartan matrix. We say that A is of finite, affine or indefinite type, if the Kac-Moody algebra $\mathfrak{g}(A)$ built from A is of finite, affine or indefinite type, respectively.*

Example 2.1.11. *Consider the $N \times N$ matrix*

$$\hat{A} = [a_{ij}]_{i,j=0}^{N-1} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}. \tag{2.2}$$

The Kac-Moody algebra $\mathfrak{g}(\hat{A})$ is affine since $\hat{A} \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0$. Note also that by removing the first row and column we get the matrix A from Example 2.1.5, which is a matrix of finite type.

This matrix can be realized on a complex vector space $\mathfrak{H}$ of dimension $2N - (N - 1) = N + 1$ as follows.

Consider the matrix

$$A^E = [a_{ij}]_{i,j=0}^{N} = \begin{pmatrix} \hat{A} & c \\ b & 2 \end{pmatrix}$$

where $c^t = (0, \dots, 0, 1) = b$. *For* $i = 0, ..., N-1$, *let* $\check{\alpha}_i$ *be the ith row of* A^E *and let* α_i *be the ith coordinate function, i.e., the linear functional defined by*

$$\alpha_i(a_0, \dots, a_N) = a_i.$$

Set

$$\Pi = \{\alpha_0, ..., \alpha_{N-1}\} \quad \text{and} \quad \check{\Pi} = \{\check{\alpha}_0, ..., \check{\alpha}_{N-1}\}.$$

Then it is easy to check that $\{\mathfrak{H} = \mathbb{C}^{N+1}, \Pi, \check{\Pi}\}$ *is a realization of* $\hat{A}$.

The Lie algebra $\mathfrak{g}(\hat{A})$ *with Chevalley generators* e_i, f_i *for* $i = 1, ..., N-1$ *and* $\mathfrak{H}$ *defined by* (2.1) *is denoted by* $\widehat{sl}_N$.

2.2 Finite vs untwisted affine

Let $A = (a_{ij})_{i,j=1}^{N-1}$ be an indecomposable generalized Cartan matrix of finite type. Then $\mathfrak{g} = \mathfrak{g}(A)$ is a finite dimensional simple Lie algebra of rank $N-1$. The finite dimensional simple Lie algebras over $\mathbb{C}$ are classified into four infinite families, A_n, $n \geq 1$, B_n, $n \geq 2$, C_n, $n \geq 3$ and D_n, $n \geq 4$, and five exceptional types, E_6, E_7, E_8, F_4 and G_2.

Let $(\mathfrak{h}, \Pi, \check{\Pi})$ be a realization of the matrix A where $\Pi = \{\alpha_1, \dots, \alpha_{N-1}\} \subseteq \mathfrak{h}^*$ and $\check{\Pi} = \{\check{\alpha}_1, ..., \check{\alpha}_{N-1}\} \subseteq \mathfrak{h}$. The subspace $\mathfrak{h}$ is called the Cartan sub-algebra of $\mathfrak{g}$. The sets Π and $\check{\Pi}$ are called the set of simple roots and simple co-roots of $\mathfrak{g}$. We can define a symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ which is invariant in the sense that $([x, y], z) = (x, [y, z])$, for all $x, y, z \in \mathfrak{g}$. The restriction of the form $(\cdot, \cdot)$ to $\mathfrak{h}$ is nondegenerate, so induces a form on $\mathfrak{h}^*$ which is determined by

$$a_{ij} = \langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \quad 1 \leq i, j \leq N-1$$

and the normalization $(\theta, \theta) = 2$ where θ is a special vector called "the highest root" (we will give this vector explicitly and explain why it is called "the highest root").

The fundamental weights of $\mathfrak{g}$ are linear functionals $\lambda_1, ..., \lambda_{N-1} \in \mathfrak{h}^*$ satisfying

$$\langle \lambda_i, \alpha_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq N-1.$$

The weight lattice of $\mathfrak{g}$ is the set

$$P = \left\{ \sum_{i=1}^{N-1} a_i\lambda_i \in \mathfrak{h}^* \middle| a_i \in \mathbb{Z} \text{ for } 1 \leq i \leq N-1 \right\}$$

and the set of dominant integral weights of g is

$$P^+ = \left\{ \sum_{i=1}^{N-1} a_i\lambda_i \;\middle|\; 0 \leq a_i \in \mathbb{Z} \right\}.$$

It is known that every irreducible finite dimensional module V for $\mathfrak{g}$ is determined up to isomorphism by its highest weight $\lambda = \sum_{i=1}^{N-1} a_i\lambda_i$ where $0 \leq a_i \in \mathbb{Z}$, for $i = 1, ..., N-1$. We now proceed to describe the structure of a $\mathfrak{g}$-module, define its highest weight and define the highest root of $\mathfrak{g}$. Any irreducible finite dimensional $\mathfrak{g}$-module V has a direct sum decomposition $V = \bigoplus_{\mu \in P} V_\mu$ into weight spaces $V_\mu = \{v \in V \mid h \cdot v = \mu(h)v, \text{ for all } h \in \mathfrak{h}\}$. Here we use $x \cdot v$ for the action of $x \in \mathfrak{g}$ on $v \in V$. Each weight space V_μ is a generalized eigenspace for the simultaneous action of the abelian Cartan sub-algebra $\mathfrak{h}$, and we set $\Pi(V) = \{\mu \in P \mid V_\mu \neq 0\}$.

The simple Lie algebra $\mathfrak{g}$ becomes a $\mathfrak{g}$-module under the adjoint action

$$ad_x(y) = x \cdot y = [x, y] \quad \text{for} \quad x, y \in \mathfrak{g}.$$

The weight space decomposition of $\mathfrak{g}$, called the Cartan decomposition, is

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{0 \neq \alpha \in P} \mathfrak{g}_\alpha,$$

where the root spaces for $0 \neq \alpha \in P$ are

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h}\}$$

and we define the set of roots by

$$\Phi = \{0 \neq \alpha \in P \mid \mathfrak{g}_\alpha \neq 0\}.$$

It is well known that $\dim(\mathfrak{g}_\alpha) = 1$ for $\alpha \in \Phi$ and since, by definition, $[h, e_i] = \alpha_i(h)e_i$ and $[h, f_i] = -\alpha_i(h)f_i$, for $1 \leq i \leq N-1$, we have $\alpha_i, -\alpha_i \in \Phi$. One has, in fact, that for any $\alpha \in \Phi$, $\alpha = \sum_{i=1}^{N-1} n_i\alpha_i$ where either all $0 \leq n_i \in \mathbb{Z}$ or $0 \geq n_i \in \mathbb{Z}$. That is, $\Phi = \Phi^+ \cup \Phi^-$ decomposes into positive and negative roots (with all simple roots positive). This gives a triangular decomposition $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$ where $\mathfrak{g}^\pm = \oplus_{\alpha\in\Phi^\pm} \mathfrak{g}_\alpha$.

Define a partial order, $\leq$, on P by

$$\lambda_1 \leq \lambda_2 \quad \text{if and only if} \quad \lambda_2 - \lambda_1 = \sum_{i=1}^{N-1} n_i \alpha_i,$$

where $0 \leq n_i \in \mathbb{Z}$ for $1 \leq i \leq N-1$.

Let V be a finite dimensional irreducible $\mathfrak{g}$-module. There is a unique weight $\lambda \in \Pi(V)$ such that $\mathfrak{g}^+ \cdot V_\lambda = 0$, $\lambda \geq \beta$ for all $\beta \in \Pi(V)$, and $\lambda = \sum_{i=1}^{N-1} a_i\lambda_i \in P^+$ ($0 \leq a_i \in \mathbb{Z}$) determines V up to isomorphism. We call λ the highest weight of V and write $V = V^\lambda$, and $\Pi^\lambda = \Pi(V^\lambda) = \{\beta \in \mathfrak{h}^* \mid V^\lambda_\beta \neq 0\} \subset P$. The highest weight θ of $\mathfrak{g}$ as a $\mathfrak{g}$-module is always a root and it is called the highest root of $\mathfrak{g}$.

The Weyl group W of $\mathfrak{g}$ is the subgroup of $GL(\mathfrak{h}^*)$ generated by the simple reflections

$$r_i(\lambda) = \lambda - \lambda(\check{\alpha}_i)\alpha_i, \quad \text{for} \quad i = 1, ..., N-1.$$

Every element $w \in W$ can be written as a word in the r_i's and the **length** $l(w)$ **of** w is defined as the number of simple reflections in a reduced word for w.

A matrix of affine type can be obtained from a matrix of finite type by adding one column and a row so that the resulting matrix has a 1-dimensional kernel. The affine Lie algebra corresponding to a matrix obtained in this way has the following construction.

Let $\hat{A} = \begin{pmatrix} 2 & b \\ c & A \end{pmatrix}$ be an $N \times N$ indecomposable generalized Cartan matrix of affine type, where A is the matrix of finite type considered at the begining of this section. There is a unique vector $u = \begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}$ such that $u > 0$, $\hat{A}u = 0$, $u_i \in \mathbb{Z}$ for $i = 0, \ldots, N-1$ and $\gcd(u_0, \ldots, u_{N-1}) = 1$. The vectors $b = (b_1, \ldots, b_{N-1})$ and $c = (c_1, \ldots, c_{N-1})^t$ are determined by

$$c_i = -(u_1 a_{i1} + \cdots + u_{N-1} a_{i,N-1}), \quad b_i = -(\hat{u}_1 a_{1i} + \cdots + \hat{u}_{N-1} a_{N-1,i}), \text{ for } 1 \leq i \leq N-1,$$

where $\hat{u}_1, \ldots, \hat{u}_{N-1}$ are the last $N-1$ components of the unique vector $\hat{u} = \begin{bmatrix} \hat{u}_0 \\ \vdots \\ \hat{u}_{N-1} \end{bmatrix}$

satisfying

$$\hat{A}^t\hat{u} = 0, \quad \hat{u}_0, \ldots, \hat{u}_{N-1} \in \mathbb{Z} \quad \text{and} \quad \gcd(\hat{u}_0, \ldots, \hat{u}_{N-1}) = 1,$$

and $a_{i1}, a_{i2} \ldots, a_{i,N-1}$ for $i = 1, \ldots N-1$ are the components of the ith row of the matrix A and $a_{1i}, a_{2i}, \ldots, a_{N-1,i}$ for $i = 1, \ldots N-1$ are the components of the ith column of A.

Let $\hat{\mathfrak{g}} = \mathfrak{g}(\hat{A})$ be the Kac-Moody algebra built from $\hat{A}$. Then $\hat{\mathfrak{g}}$ is an affine Lie algebra with realization $(\mathfrak{H}, \Psi, \hat{\Psi})$ where $\dim \mathfrak{H} = N + 1$. The number

$$h = \sum_{i=0}^{N-1} u_i$$

is called the Coxeter number and

$$\check{h} = \sum_{i=0}^{N-1} \check{u}_i$$

is called the **dual Coxeter number**. The Lie algebra $\hat{\mathfrak{g}}$ has a one dimensional center spanned by

$$c = \sum_{i=0}^{N-1} \check{u}_i \check{\alpha}_i.$$

The element c is called the **canonical central element**. The highest root of the finite dimensional Lie algebra $\mathfrak{g}$ is explicitly given by

$$\theta = \sum_{i=1}^{N-1} u_i \alpha_i.$$

The realization of $\hat{\mathfrak{g}}$, $(\mathfrak{H}, \Psi, \check{\Psi})$, can be constructed so that $\mathfrak{h} \subseteq \mathfrak{H}$, $\Pi \subseteq \Psi$ and $\check{\Pi} \subseteq \check{\Psi}$. This can be done as follows.

There exists an element in $d \in \mathfrak{H}$ satisfying the conditions $\alpha_0(d) = 1$ and $\alpha_i(d) = 0$ for $i = 1, \ldots, N-1$. The vector d is uniquely determined up to multiple of c and $B = \{c, \check{\alpha}_1, \ldots, \check{\alpha}_{N-1}, d\}$ is a basis for $\mathfrak{H}$.

Let $d^* \in \mathfrak{H}^*$ be the linear functional dual to d with respect to the basis B. This linear functional is called the **null root** of $\hat{\mathfrak{g}}$ and it is given explicitly by

$$d^* = \sum_{i=0}^{N-1} u_i \alpha_i \in \mathfrak{H}^*.$$

Let $c^* \in \mathfrak{H}^*$ be the element dual to the canonical central element c with respect to the basis B. Then we have $c^*(c) = 1$, $c^*(d) = 0$ and $d^*(c) = 0$.

We take $\Psi = \{\alpha_0, \alpha_1, \ldots, \alpha_{N-1}\}$ and $\check{\Psi} = \{\check{\alpha}_0, \check{\alpha}_1, \ldots, \check{\alpha}_{N-1}\}$. The sets Ψ and $\check{\Psi}$ are called the set of simple roots and simple co-roots of $\hat{\mathfrak{g}}$.

We also identify linear functionals in $\mathfrak{h}^*$ with linear functionals in $\mathfrak{H}^*$ by having the same values on $\mathfrak{h}$ and being zero on α_0 and d.

The fundamental weights of $\hat{\mathfrak{g}}$ are linear functionals $\Lambda_0, \ldots, \Lambda_{N-1} \in \mathfrak{h}^*$ satisfying

$$\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}, \quad 0 \leq i, j \leq N-1,$$

and it can be checked that the fundamental weights are given by

$$\Lambda_0 = c^*, \quad \Lambda_i = u_i \frac{(\alpha_i, \alpha_i)}{2} c^* + \lambda_i, \quad \text{for} \quad 1 \leq i \leq N-1,$$

where $\Lambda_0 = c^* + rd^*$ is normalized by taking $r = 0$ and $\lambda_1, \ldots, \lambda_{N-1}$ are the weights for the finite dimensional Lie algebra $\mathfrak{g}$.

The set of integral weights for $\hat{\mathfrak{g}}$ is defined by

$$\hat{P} = \left\{ \Lambda = \sum_{i=0}^{N-1} n_i \Lambda_i \,\middle|\, n_i \in \mathbb{Z} \right\}.$$

The set of dominant integral weights for $\hat{\mathfrak{g}}$ is defined by

$$\hat{P}^+ = \left\{ \Lambda = \sum_{i=0}^{N-1} n_i \Lambda_i \,\middle|\, 0 \leq n_i \in \mathbb{Z} \right\}.$$

We can also define a partial order $\leq$ on $\mathfrak{H}^*$ by

$$\Lambda \leq \Lambda' \text{ if and only if } \Lambda' - \Lambda = \sum_{i=0}^{N-1} a_i \alpha_i, \quad \text{for } 0 \leq a_i \in \mathbb{Z}.$$

The affine Lie algebras are classified into 2 types, twisted and untwisted. For the untwisted affine Lie algebras $\hat{\mathfrak{g}}$ we get an isomorphism of Lie algebras

$$\hat{\mathfrak{g}} \cong \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

where $\mathfrak{g}$ is the finite dimensional Lie algebra and the bracket on $\mathbb{C}[t,t^{-1}]\otimes\mathfrak{g}\oplus\mathbb{C}c\oplus\mathbb{C}d$ is defined by

$$[x(m),y(n)]=[x,y](m+n)+m\delta_{m+n,0}(x,y)c,$$

$$[d,x(m)]=mx(m),\quad \text{and}\quad [c,x(m)]=[c,d]=0,$$

where $x(m)=t^m\otimes x$, and $(\cdot,\cdot)$ is the normalized invariant symmetric bilinear form from $\mathfrak{g}$, $x,y\in\mathfrak{g}$. The restriction of $(\cdot,\cdot)$ to $\mathfrak{h}$ is non-degenerate, so it induces a form on $\mathfrak{h}^*$. The normalization is taken so that $(\theta,\theta)=2$. We say that $\hat{\mathfrak{g}}$ is of type $X_n^{(1)}$ if $\mathfrak{g}$ is of type X_n, where $X=A,B,C,D,E,F$ or G.

We can extend the symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ to $\hat{\mathfrak{g}}$ by

$$(x(m),y(n))=\delta_{m+n,0}(x,y),\quad (x(m),c)=(y(n),d)=(c,c)=(d,d)=0,$$

and

$$(c,d)=1.$$

The Weyl group $\widehat{W}$ of $\hat{\mathfrak{g}}$ is the subgroup of $GL(\mathfrak{H}^*)$ generated by the simple reflections

$$r_i(\Lambda)=\Lambda-\Lambda(\check{\alpha}_i)\alpha_i,\quad \text{for}\quad i=0,...,N-1,$$

and $W=\langle r_i\mid 1\le i\le N-1\rangle$ is a subgroup of $\widehat{W}$. For each $0\le k\in\mathbb{Z}$ the affine Weyl group $\widehat{W}$ acts on the weight lattice P of $\mathfrak{g}$ with the usual action of the simple reflections of W and with

$$r_0(\beta)=r_\theta(\beta)+(k+\check{h})\theta, \tag{2.3}$$

where $r_\theta(\lambda)=\lambda-\dfrac{2(\lambda,\theta)}{(\theta,\theta)}\theta$ is the reflection with respect to the highest root θ and $\check{h}$ is the dual Coxeter number. The set of weights of level k of $\mathfrak{g}$ is the set

$$P_k^+=\left\{\sum_{j=1}^{N-1}a_j\lambda_j\in P^+\;\middle|\;\sum_{j=1}^{N-1}a_j\le k\right\},$$

and the fundamental region for the action of $\widehat{W}$ on P is $P^+_{k+\check{h}}$.

The representation theory of $\hat{\mathfrak{g}}$ is very similar to the representation theory of $\mathfrak{g}$ if we restrict only to $\hat{\mathfrak{g}}$-modules V that satisfy the following, where

$$V_\Lambda=\{v\in V\mid h\cdot v=\Lambda(h)v,\quad \text{for all}\quad h\in\mathfrak{H}\},\qquad \Pi(V)=\{\Lambda\in\mathfrak{H}^*\mid V_\Lambda\ne 0\}$$

and

$$D(\lambda) = \{\mu \in \mathfrak{H}^* \mid \mu \leq \lambda\}.$$

i) V is $\mathfrak{H}$-diagonalizable, i.e., $V = \bigoplus_{\Lambda \in \mathfrak{H}^*} V_\Lambda$.

ii) $\dim V_\Lambda < \infty$, for all $\Lambda \in \mathfrak{H}^*$.

iii) There exits a finite number of elements $\beta_1, \ldots, \beta_s$ so that $\Pi(V) \subseteq \cup_{i=1}^s D(\beta_i)$.

As a consequence, every $\hat{\mathfrak{g}}$-module V satisfying the above conditions has a highest weight $\Lambda \in \hat{P}$ and if V is irreducible then it would be determined, up to isomorphism, by its highest weight Λ. We use the notation V^Λ to denote the irreducible highest weight module for $\hat{\mathfrak{g}}$. The canonical central element acts on V^Λ as a scalar k and we have

$$k = \Lambda(c) = \sum_{i=0}^{N-1} \Lambda_i(c) = \sum_{i=0}^{N-1} u_i \frac{(\alpha_i, \alpha_i)}{2}$$

so for a fixed k, there are only finitely many $\Lambda \in \hat{P}^+$ with $\Lambda(c) = k$.

Example 2.2.1. *The untwisted affine algebra $\widehat{sl}_N$ defined in Example 2.1.11 has simple roots*

$$\alpha_0, \ldots, \alpha_{N-1}.$$

The vector $u = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ satisfies $\hat{A} \cdot u = 0$ where $\hat{A}$ is the matrix given in (2.2) *and since $\hat{A}$ is symmetric we also have $\hat{A}^t \cdot u = 0$. Then the canonical central element c and the null root d^* are given by*

$$c = \sum_{i=0}^{N-1} \check{\alpha}_i \quad \text{and} \quad d^* = \sum_{i=0}^{N-1} \alpha_i.$$

The fundamental weights are given by the equations

$$\Lambda_0 = c^*, \quad \Lambda_i = c^* + \lambda_i, \quad \text{for} \quad 1 \leq i \leq N-1, \tag{2.4}$$

where $\lambda_1, \ldots, \lambda_{N-1}$ are the fundamental weights for the Lie algebra $\mathfrak{g} = sl_N$.

Every irreducible highest weight module $\hat{V}$ for $\widehat{sl}_N$ on level k is determined, up to isomorphism, by a weight $\Lambda = \sum_{i=0}^{N-1} n_i \Lambda_i$ satisfying the condition $k = \sum_{i=0}^{N-1} n_i$.

From (2.4) *we see that*

$$\Lambda = n_0 c^* + \sum_{i=1}^{N-1} (n_i c^* + n_i \lambda_i) = k c^* + \sum_{i=1}^{N-1} n_i \lambda_i,$$

where $\sum_{i=1}^{N-1} n_i \leq k$.

The Coxeter and dual Coxeter numbers of $\hat{\mathfrak{g}}$ *are* $h = \check{h} = N$ *and the level* k *action of the affine reflection* r_0 *on the weight lattice* P *of* $\mathfrak{g}$ *is given by*

$$r_0(\lambda) = r_\theta(\lambda) + (k + \hat{h})\theta,$$

where $\theta = \sum_{i=1}^{N-1} \alpha_i$ *is the highest root of* $\mathfrak{g} = sl_N$.

2.3 Fusion algebras

Let us begin with the definition of fusion algebra, due to J. Fuchs [Fu1]. (Also see [Fe].)

Definition 2.3.1. *A fusion algebra is a finite dimensional commutative associative algebra* $\mathcal{F}$ *over* $\mathbb{Q}$ *satisfying the following.*

1) There is a distinguished basis

$$B = \{x_a \mid a \in A\}, \quad \text{for some finite index set } A,$$

so that the product of the basis elements is given by

$$x_a \cdot x_b = \sum_{c \in A} N_{a,b}^c x_c,$$

where $0 \leq N_{a,b}^c \in \mathbb{Z}$.

2) There is an element $\omega \in A$ *such that the map*

$$\mathcal{C}(x_a) = \sum_{b \in A} N_{a,b}^\omega x_b \tag{2.5}$$

is an involution. That is, the matrix $C = (C_{ab})_{a,b \in A}$ *for* $\mathcal{C}$*, where* $C_{ab} = N_{a,b}^\omega$*, satisfies the equation* $C^2 = I$ *(the identity matrix). The map* $\mathcal{C}$ *is called the* ***conjugation map****.*

We have several consequences from the definition. Since the entries of C are non-negative integers and $C^2 = I$, then it follows that either $C = I$ or C is a permutation matrix of order 2. Hence, there is a permutation $\sigma : A \to A$ satisfying $\sigma^2 = 1$ and so that

$$C_{a,b} = \delta_{a,\sigma(b)}.$$

The conjugation map $\mathcal{C} : \mathcal{F} \to \mathcal{F}$ defined in (2.5), is then given by

$$\mathcal{C}(x_a) = \sum_{b \in A} \delta_{a,\sigma(b)} x_b = x_{\sigma(a)}$$

and since it is an automorphism, we also get

$$\begin{aligned} x_{\sigma(a)} \cdot x_{\sigma(b)} &= \mathcal{C}(x_a)\mathcal{C}(x_b) = \mathcal{C}(x_a \cdot x_b) = \mathcal{C}\left(\sum_{c \in A} N_{a,b}^c x_c\right) \\ &= \sum_{c \in A} N_{a,b}^c \mathcal{C}(x_c) = \sum_{c \in A} N_{a,b}^c x_{\sigma(c)}. \end{aligned}$$

Therefore we get

$$N_{a,b}^c = N_{\sigma(a),\sigma(b)}^{\sigma(c)}.$$

Set

$$N_{a,b,c} = N_{a,b}^{\sigma(c)}.$$

It follows from commutativity and associativity of $\mathcal{F}$ that the constants $N_{a,b,c}$ are totally symmetric in a, b and c. Then we also get

$$N_{\omega,b}^c = N_{\omega,b,\sigma(c)} = N_{b,\sigma(c),\omega} = N_{b,\sigma(c)}^{\sigma(\omega)} = N_{\sigma(b),c}^{\omega} = C_{\sigma(b),c} = \delta_{b,c},$$

hence

$$x_\omega x_b = \sum_{c \in A} N_{\omega,b}^c x_c = \sum_{c \in A} \delta_{b,c} x_c = x_b.$$

So x_ω is a multiplicative identity and $\sigma(\omega) = \omega$.

Example 2.3.2. *Consider a 3-dimensional algebra $\mathcal{F}$ over $\mathbb{Q}$ with basis $B = \{x_0, x_1, x_2\}$ and product given by the table (where the blank entries are determined by commutativity.)*

	x_0	x_1	x_2
x_0	x_0	x_1	x_2
x_1		x_0	x_2
x_2			$x_0 + x_1$

It can be checked that the product defined by this table is associative with identity element x_0, and the conjugation map $\mathcal{C}$ is the identity map with $C = I$. Therefore $\mathcal{F}$ is a fusion algebra.

2.4 Fusion algebras from affine Lie algebras

In this section we discuss the decomposition of the tensor product of two irreducible finite dimensional modules for a finite dimensional Lie algebra $\mathfrak{g}$ and use it to define a "level k truncated tensor product" of two irreducible highest weight modules for the untwisted affine Lie algebra $\hat{\mathfrak{g}}$. This truncated product defines a fusion algebra with basis given by irreducible highest weight modules of $\hat{\mathfrak{g}}$ on level k.

Let $\mathfrak{g}$ be a finite dimensional algebra and let V^λ and V^μ be two irreducible finite dimensional modules for $\mathfrak{g}$. The tensor product $V^\lambda \otimes V^\mu$ is completely reducible. Therefore it can be expressed as a direct sum of irreducible modules

$$V^\lambda \otimes V^\mu = \bigoplus_{\nu \in P^+} Mult_{\lambda,\mu}(\nu) V^\nu.$$

The coefficients $Mult_{\lambda,\mu}(\nu)$ can be computed by means of the Racah-Speiser algorithm which can be described as follows. For any $\nu \in P^+$ we have

$$Mult_{\lambda,\mu}(\nu) = \sum_{w \in W} (-1)^{l(w)} Mult_\lambda(w(\nu + \rho) - \mu - \rho), \tag{2.6}$$

where $Mult_\lambda(\beta) = dim(V^\lambda_\beta)$ and $\rho = \sum_{i=1}^{N-1} \lambda_i$. This formula yields the following geometric algorithm.

Step 1.

Shift the weight diagram Π^λ of V^λ by adding $\mu + \rho$ to each weight.

Step 2.

Use the finite Weyl group to move all shifted weights $\Pi^\lambda + \mu + \rho$ into the dominant chamber, P^+, where they accumulate as an alternating sum of inner multiplicities of V^λ, adding if the required w is even, subtracting if it is odd.

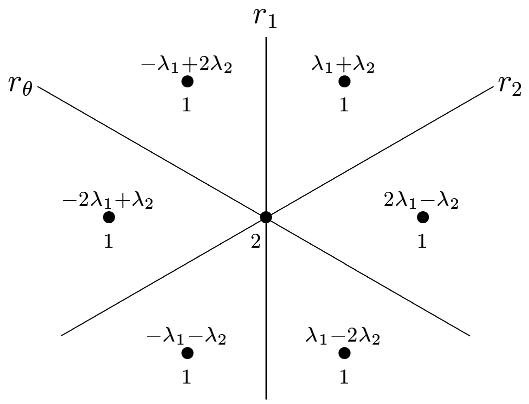
Step 3.

The resulting pattern of numbers will be non-negative integers, zero if on a chamber wall, and after shifting the pattern back by subtracting ρ we get the tensor product multiplicities.

Example 2.4.1. *Consider the modules V^λ and V^μ of sl_3 where $\lambda = \lambda_1 + \lambda_2$ and $\mu = 2\lambda_1$. The weight decomposition of V^λ is given by*

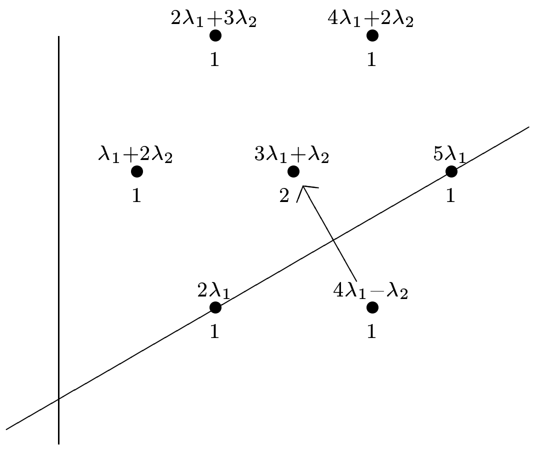
$$V^\lambda = V_{\lambda_1+\lambda_2} \oplus V_{2\lambda_1-\lambda_2} \oplus V_{-\lambda_1+2\lambda_2} \oplus 2V_0 \oplus V_{\lambda_1-2\lambda_2} \oplus V_{-2\lambda_1+\lambda_2} \oplus V_{-\lambda_1-\lambda_2}.$$

So the weight diagram of V^λ is given by

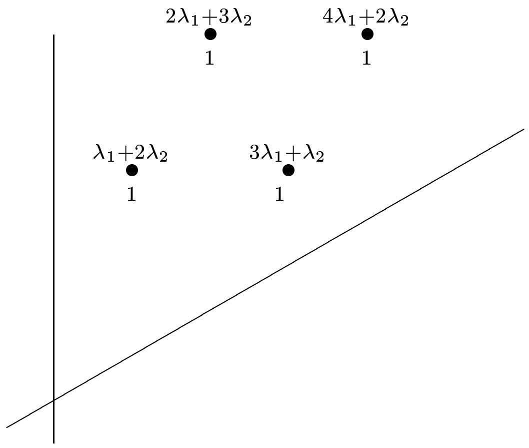


where the number below each dot is the inner multiplicity, or the dimension of the weight space, and the two lines labeled r_1 and r_2 are the fixed lines of the simple reflections.

Now, adding $\mu+\rho = 2\lambda_1+(\lambda_1+\lambda_2) = 3\lambda_1+\lambda_2$ to all weights in the above diagram, we get



where the arrow indicates that the weight outside the fundamental chamber gets reflected onto the weight inside the fundamental chamber, reducing its multiplicity by 1, and since the weights on the chamber wall do not count for the tensor product, we are left with the pattern



Subtracting $\rho = \lambda_1 + \lambda_2$ *from all these weights we get the tensor product decomposition of* $V^{\lambda_1+\lambda_2} \otimes V^{2\lambda_1}$*, that is*

$$V^{\lambda_1+\lambda_2} \otimes V^{2\lambda_1} = V^{3\lambda_1+\lambda_2} \oplus V^{\lambda_1+2\lambda_2} \oplus V^{2\lambda_1} \oplus V^{\lambda_2}.$$

The Racah-Speiser algorithm was modified by Kac and Walton for computing the fusion coefficients $N^{(k)\nu}_{\mu,\lambda}$ for $\lambda, \mu, \nu \in P^+_k$, as follows. Under the level k action of $\widehat{W}$ on P we have

$$N^{(k)\nu}_{\mu,\lambda} = \sum_{w \in \widehat{W}} (-1)^{l(w)} Mult_\lambda(w(\nu+\rho) - \mu - \rho),$$

where $Mult_\lambda(\beta) = dim(V^\lambda_\beta)$ and $\rho = \sum_{i=1}^{N-1} \lambda_i$. This formula now involving a summation over the affine Weyl group, $\widehat{W}$, also yields a geometric algorithm as follows.

Step 1.

Shift the weight diagram of V^λ by adding $\mu + \rho$ to each weight.

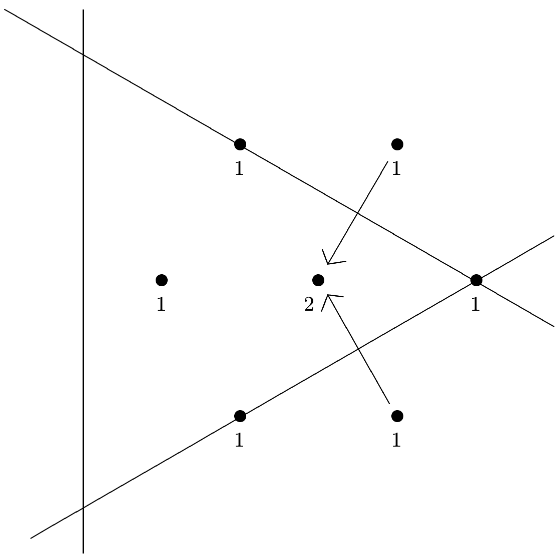
Step 2.

Use the level k action of the affine Weyl group $\widehat{W}$ on P to move all shifted weights into the fundamental domain, P_k^+, bounded by the reflection walls of all simple reflections r_i, for $1 \leq i \leq N-1$, and of the affine reflection r_0. That affine reflection is a hyperplane perpendicular to θ going through the point $(k + \check{h})\theta$. The reflected weights counted with inner multiplicities accumulate as an alternating sum of inner multiplicities of V^λ, adding if the required w is even, subtracting if it is odd.

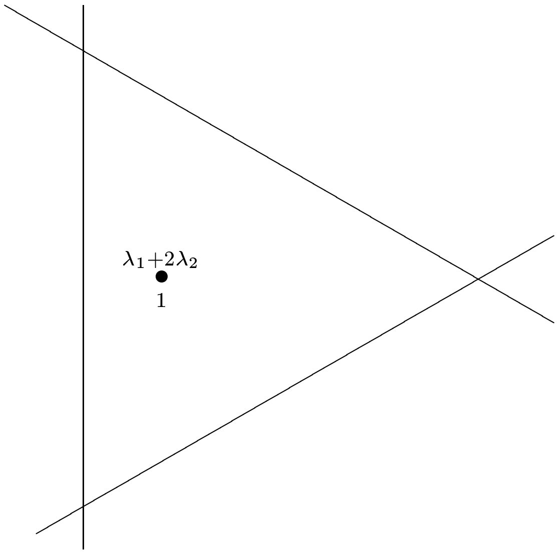
Step 3.

The resulting pattern of numbers will be non-negative integers, zero if on a reflection wall, and after shifting the pattern back by subtracting ρ, we get the level k fusion product multiplicities.

Example 2.4.2. *Consider the modules V^λ and V^μ of sl_3 where $\lambda = \lambda_1 + \lambda_2$ and $\mu = 2\lambda_1$ and let $k = 2$. The weight decomposition and weight diagram of V^λ is given in the previous example. Adding $\mu + \rho = 2\lambda_1 + \lambda_1 + \lambda_2 = 3\lambda_1 + \lambda_2$ to all weights in its diagram, we get*



where the arrows indicate each of the weights that are outside the affine fundamental domain, and get reflected onto a weight inside the affine fundamental domain. In this case, 2 outside weights get reflected onto the weight $3\lambda_1 + \lambda_2$*, reducing its multiplicity by 2, and since the weights on the reflection wall do not contribute to the fusion product, we are left with the pattern*



Subtracting $\rho = \lambda_1 + \lambda_2$ *from this weight we get the fusion product*

$$V^{\lambda_1+\lambda_2} \otimes_2 V^{2\lambda_1} = V^{\lambda_2}.$$

2.5 Notation for the rest of the thesis

Let $\hat{\mathfrak{g}}$ be the affine algebra of type $A^{(1)}_{N-1}$ built from $\mathfrak{g} = sl_N$. Let $\lambda_1, \lambda_2, ..., \lambda_{N-1}$ denote the fundamental weights of $\mathfrak{g}$ and let:

$$P^+ = \left\{ \sum_{j=1}^{N-1} a_j \lambda_j \;\middle|\; 0 \leq a_j \in \mathbb{Z} \right\} \tag{2.7}$$

denote the dominant integral weights of $\mathfrak{g}$. The irreducible modules V^Λ for $\hat{\mathfrak{g}}$ of level k are indexed by dominant highest weights $\Lambda = kc + \lambda$ where λ is in

$$P^+_k = \left\{ \sum_{j=1}^{N-1} a_j \lambda_j \in P^+ \;\middle|\; \sum_{j=1}^{N-1} a_j \leq k \right\}.$$

However, for a fixed level k, we will denote these modules by V^λ instead, where $\lambda \in P_k^+$. These weights are in one-to-one correspondence with the set of N-tuples $(a_0, a_1, ..., a_{N-1})$ whose sum is k, where $a_0 = k - \sum_{j=1}^{N-1} a_j$.

Let G be the group $\mathbb{Z}_N^k$ and let S_k act on it by permuting the k-tuples, so every orbit of $\mathbb{Z}_N^k$ under this action has a unique standard representative in the form

$$((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0^{a_0})$$

where the exponent indicates the number of repetitions of the base.

We get a one-to-one correspondence between S_k-orbits of $\mathbb{Z}_N^k$ and N-tuples whose sum is k, by:

$$((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0^{a_0}) \longleftrightarrow (a_0, a_1, ..., a_{N-1}) \tag{2.8}$$

therefore we get a correspondence between weights $\lambda \in P_k^+$ and orbits of $\mathbb{Z}_N^k$ under the described action of S_k. The orbit corresponding to λ will be denoted by $[\lambda]$ and the correspondence is given by

$$\lambda = \sum_{j=1}^{N-1} a_j \lambda_j \in P_k^+ \longmapsto [\lambda] = [((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0^{a_0})], \tag{2.9}$$

where $a_0 = k - \sum_{j=1}^{N-1} a_j$ and $[((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0^{a_0})]$ denotes the orbit of $\mathbb{Z}_N^k$ whose standard representative is $((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0^{a_0})$.

A partition is a finite sequence of non-negative integers $(\mu_1, \dots, \mu_n, \dots)$ so that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \dots$. The length of the partition μ, $l(\mu)$, is the number of non-zero μ_i's.

There is also a map between partitions of length at most N and dominant integral weights of A_{N-1} given by:

$$(\mu_1, ..., \mu_N) \longmapsto \lambda = \sum_{j=1}^{N-1} (\mu_j - \mu_{j+1}) \lambda_j \in P^+ \tag{2.10}$$

and if $\mu_1 - \mu_N \leq k$ then $\lambda \in P_k^+$.

Note that this map is not one-to-one since given a partition $(\mu) = (\mu_1, ..., \mu_N)$ with $\mu_N \neq 0$, then the partitions (μ) and $(\mu_1 - \mu_N, \mu_2 - \mu_N, ..., \mu_{N-1} - \mu_N)$ lead to the same weight. However, this correspondence has a right inverse which can be described as follows. If $\lambda = \sum_{j=1}^{N-1} a_j \lambda_j \in P^+$ then the partition associated to the weight λ, which will be denoted by (λ), is given by the map

$$\lambda = \sum_{j=1}^{N-1} a_j \lambda_j \longmapsto (\lambda) = \left(\sum_{j=1}^{N-1} a_j, \sum_{j=2}^{N-1} a_j, ..., a_{N-1} \right), \tag{2.11}$$

and if $\lambda \in P_k^+$ then (λ) has largest part at most k.

Another way of defining (2.10) is by defining an equivalence relation $\sim$ on partitions of length at most N by

$$(\mu_1, ..., \mu_N) \sim (\nu_1, ..., \nu_N) \text{ if and only if } \mu_i - \mu_{i+1} = \nu_i - \nu_{i+1} \text{ for } 1 \leq i \leq N-1. \tag{2.12}$$

The equivalence class for $(\mu_1, ..., \mu_N)$ has a unique representative in the set of partitions of length at most $N-1$ given by $(\mu_1 - \mu_N, \mu_2 - \mu_N, ..., \mu_{N-1} - \mu_N)$ and it is clear from (2.10) that both can be used to get the weight associated to $(\mu_1, ..., \mu_N)$. Furthermore two such partitions (μ) and (μ') are in the same equivalence class under $\sim$ if and only if $\mu'_i = \mu_i + c$ for some integer c and for $1 \leq i \leq N$.

It is clear that the left hand side of (2.8) is also a partition, as the sequence is non-increasing, which will be proved to be the conjugate of the partition on the right hand side of (2.11). Combining (2.9) and (2.11) gives a map from S_k-orbits of $\mathbb{Z}_N^k$ to partitions of length at most $N-1$ with largest part at most k. And from (2.9) and (2.10) we get a map from partitions $(\mu_1, ..., \mu_N)$ of length at most N with $\mu_1 - \mu_N \leq k$ to S_k-orbits of $\mathbb{Z}_N^k$. This will be useful in proving that for specific λ to be described in section 4 below, the operation among S_k-orbits defined by Feingold and Weiner [FW], matches level k fusion Pieri rules for computing fusion coefficients $N_{\lambda,\mu}^{(k)\nu}$. We will prove that these specific weights include generators of the fusion ring of type A_{N-1} of level k, $\mathcal{F}(A_{N-1}, k)$. Therefore, using the Jacobi-Trudi determinant, a simple iteration allows one to compute the rest of the structure constants $N_{\lambda,\mu}^{(k)\nu}$ for arbitrary μ and ν.

This describes a new method of computing type A coefficients which uses only arithmetic in $\mathbb{Z}_N^k$ and an iteration to obtain the whole multiplication table for $\mathcal{F}(A_{N-1}, k)$.

Chapter 3

Orbits of $\mathbb{Z}_N^k$ under the action of S_k

In this chapter we follow Feingold and Weiner [FW]. Let G be the group $\mathbb{Z}_N^k$, and for each N-tuple of nonnegative integers

$$(a_0, a_1, ..., a_{N-1}) \qquad \text{such that} \qquad a_0 + a_1 + ... + a_{N-1} = k$$

we define the subset of G

$$[(a_0, a_1, ..., a_{N-1})] = \left\{ x \in \mathbb{Z}_N^k \mid j \text{ occurs } a_j \text{ times in } x,\ 0 \le j \le N-1 \right\}. \tag{3.1}$$

Then G is a disjoint union of these subsets.

Note that the symmetric group S_k acts on G by permuting k-tuples, the **set of orbits under this action**, $\mathcal{O} = \mathcal{O}(N, k)$, consists of the subsets (3.1) defined above, and each orbit contains a unique representative in **standard form**

$$((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0^{a_0}) \tag{3.2}$$

where the exponent indicates the number of repetitions of the base.

Notation 3.0.1. *Given $x \in \mathbb{Z}_N^k$, we will denote the **orbit of x** by $[x]$ and the **representative in standard form of this orbit** will be denoted by $\hat{x}$.*

For orbits $[a]$,$[b]$ and $[c]$, we define the set:

$$T([a], [b], [c]) = \{(x, y, z) \in [a] \times [b] \times [c] \mid x + y = z\}.$$

Note that $\sigma \in S_k$ acts on $(x, y, z) \in T([a], [b], [c])$ by $\sigma(x, y, z) = (\sigma x, \sigma y, \sigma z)$.

Definition 3.0.2. *Denote by* $\mathbf{M}^{(\mathbf{k})[\mathbf{c}]}_{[\mathbf{a}],[\mathbf{b}]}$ ***the number of*** $\mathbf{S_k}$***-orbits of*** $\mathbf{T}([\mathbf{a}],[\mathbf{b}],[\mathbf{c}])$.

This suggests that we could use these numbers as the structure constants of an algebra (depending on N and k and over any field of characteristic 0) with basis $\mathcal{O}$, by defining a bilinear product as follows:

$$[a] \times [b] = \sum_{c \in \mathcal{O}} M^{(k)[c]}_{[a],[b]} [c]. \tag{3.3}$$

In [FW], it was shown that for $N = 2$ and all $k \geq 1$, this product coincides with the product in the associative fusion algebra $\mathcal{F}(A_1, k)$, but for $N = 3$ and all $k \geq 1$ it was shown that

$$M^{(k)[\nu]}_{[\mu],[\lambda]} = \binom{N^{(k)\nu}_{\mu,\lambda} + 1}{2}. \tag{3.4}$$

Now we proceed to describe how to compute the product of two S_k-orbits of $\mathbb{Z}_N^k$. This description will be useful in proving some theorems in later chapters.

Definition 3.0.3. *Let* $[a]$*,* $[b] \in \mathcal{O}$*, and assume that* $[b] = \{y_1, ..., y_t\}$*. For* $1 \leq i \leq t$ *set:*

$$z_i = \hat{a} + y_i. \tag{3.5}$$

We say that the equation $z_j = \hat{a} + y_j$ *in the list* (3.5) *is* ***redundant****, if for some* $i < j$ *and* $\sigma \in S_k$ *we have*

$$\sigma\hat{a} = \hat{a},\ \sigma y_j = y_i \text{ and } \sigma z_j = z_i,$$

that is, if the triples $(\hat{a}, y_i, z_i)$ *and* $(\hat{a}, y_j, z_j)$ *are in the same* S_k*-orbit of* $T([a],[b],[z_i])$.

Now we can describe how the product of two orbits can be computed. Let $[a]$, $[b] \in \mathcal{O}$ and fix a representative from the orbit $[a]$, say the representative in standard form, $\hat{a}$, and assume that $[b] = \{y_1, ..., y_t\}$. For every $y_i \in [b]$, set

$$z_i = \hat{a} + y_i, \qquad 1 \leq i \leq t.$$

Remove all redundant equations from this list, and without loss of generality, assume that after removing all redundancies, we are left with the first s equations, for some $s \leq t$. That is, the list:

$$z_i = \hat{a} + y_i, \qquad 1 \leq i \leq s,$$

has no redundancies. Then

$$[a] \times [b] = [z_1] + [z_2] + ... + [z_s].$$

Note that several z_i's could be in the same orbit, and for every $[c] \in \mathcal{O}$,

$$M^{(k)[c]}_{[a],[b]} = Card\{1 \leq i \leq s \mid z_i \in [c]\}. \tag{3.6}$$

Remark 3.0.4. *From Equation* (3.6)*, we also get that* $M^{(k)[c]}_{[a],[b]}$ *can be computed by removing all redundancies from the list of equations*

$$z = \hat{a} + y,$$

where $y \in [b]$ *and* $z \in [c]$*.*

Example 3.0.5. *Let* $a = (2, 1, 0)$ *and* $b = (1, 1, 0)$ *be elements in* $\mathbb{Z}_3^3$*. Then we have that* $\hat{a} = (2, 1, 0)$ *and* $[b] = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$*. Now to compute* $[a] \times [b]$*, we remove all redundancies from the list:*

$$(2, 1, 0) + (1, 1, 0) = (0, 2, 0),$$
$$(2, 1, 0) + (1, 0, 1) = (0, 1, 1),$$
$$(2, 1, 0) + (0, 1, 1) = (2, 2, 1).$$

Since there are no redundant equations in that list, we get that:

$$[(2, 1, 0)] \times [(1, 1, 0)] = [(2, 2, 1)] + [(1, 1, 0)] + [(2, 0, 0)]. \tag{3.7}$$

Example 3.0.6. *Now let* $a = (2, 2, 1)$ *and* $b = (1, 0, 0)$ *be elements in* $\mathbb{Z}_3^3$*. Then we have that* $\hat{a} = (2, 2, 1)$ *and* $[b] = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$*. To compute* $[a] \times [b]$*, we remove all redundancies from the list:*

$$(2, 2, 1) + (1, 0, 0) = (0, 2, 1),$$
$$(2, 2, 1) + (0, 1, 0) = (2, 0, 1),$$
$$(2, 2, 1) + (0, 0, 1) = (2, 2, 2).$$

We can see that the second equation is redundant, so we can remove it from the list and we get

$$[(2, 2, 1)] \times [(1, 0, 0)] = [(2, 2, 2)] + [(2, 1, 0)].$$

Example 3.0.7. *Now let* $a = (3,2,1)$ *and* $b = (1,1,0)$ *be elements in* $\mathbb{Z}_4^3$. *Then we have that* $\hat{a} = (3,2,1)$ *and* $[b] = \{(1,1,0),(1,0,1),(0,1,1)\}$. *To compute* $[a] \times [b]$, *we remove all redundancies from the list:*

$$(3,2,1) + (1,1,0) = (0,3,1),$$
$$(3,2,1) + (1,0,1) = (0,2,2),$$
$$(3,2,1) + (0,1,1) = (3,3,2).$$

There are no redundant equations in this list, so we get:

$$[(3,2,1)] \times [(1,1,0)] = [(3,1,0)] + [(2,2,0)] + [(3,3,2)]. \tag{3.8}$$

In our next example we show how this product of orbits fails to be associative.

Example 3.0.8. *Let* $a = (2,1,0)$, $b = (1,1,0)$ *and* $c = (1,0,0)$ *in* $\mathbb{Z}_3^3$. *An easy calculation using the results of Examples 3.0.5 and 3.0.6 shows that:*

$$([a] \times [b]) \times [c] = [(2,2,2)] + [(1,1,1)] + 3[(2,1,0)] + [(0,0,0)]$$

and

$$[a] \times ([b] \times [c]) = [(2,2,2)] + [(1,1,1)] + 4[(2,1,0)] + [(0,0,0)].$$

Note that multiplicities of $[(2,1,0)]$ *do not match.*

Our aim is to prove, once we establish the connection between Young diagrams and orbits, that the operation (3.3) matches fusion Pieri rules when $[b]$ is the orbit corresponding to a weight of a certain form.

In Examples 3.0.5, 3.0.6 and 3.0.7 we computed the product of two orbits where one of the orbits consisted of only zeroes and ones and we always got multiplicity 1 for every one of the orbits in the product. The next lemma, which is essential in meeting our aim, shows that this is always true. (In general, multiplicities could be higher than one. See Example 3.0.11 at the end of this chapter.)

Theorem 3.0.9. *Let* $[a],[c] \in \mathcal{O}$ *and* $[b] = [(1^m, 0^{k-m})]$ *for some* $m \leq k$. *Suppose that* $M^{(k)[c]}_{[a],[b]} \neq 0$ *then* $M^{(k)[c]}_{[a],[b]} = 1$.

Proof. Let's assume that the representative in standard form, $\hat{a}$, of the the orbit $[a]$ is

$$\hat{a} = ((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0^{a_0}) . \tag{3.9}$$

The following claim is the key of the proof.

Claim: If $z \in [c]$ and $y \in [b]$ satisfy the equation

$$z = \hat{a} + y, \tag{3.10}$$

then there exists $\sigma \in S_k$ so that $\sigma\hat{a} = \hat{a}$,

$$\sigma y = (1^{m_{N-1}}, 0^{a_{N-1}-m_{N-1}}, ..., 1^{m_1}, 0^{a_1-m_1}, 1^{m_0}, 0^{a_0-m_0})$$

and

$$\sigma z = \left(0^{m_{N-1}}, (N-1)^{a_{N-1}-m_{N-1}+m_{N-2}}, ..., 1^{a_1-m_1+m_0}, 0^{a_0-m_0}\right),$$

where m_0, m_1,...,m_{N-1} are integers such that $0 \leq m_i \leq a_i$, for $0 \leq i \leq N-1$ and $m_0 + m_1 + ... + m_{N-1} = m$.

Proof of the claim:

Consider the bijection:

$$\begin{aligned} \alpha : \mathbb{Z}_N^k &\longrightarrow \mathbb{Z}_N^{a_{N-1}} \times ... \times \mathbb{Z}_N^{a_1} \times \mathbb{Z}_N^{a_0} \\ p &\longmapsto (p_{N-1}, ..., p_1, p_0) \end{aligned}$$

where p_{N-1} consists of the first a_{N-1} entries of p, p_{N-2} contains the next a_{N-2} entries of p and so on, until finally p_0 contains the last a_0 entries of p. Let $(y_{N-1}, ..., y_1, y_0) \in \mathbb{Z}_N^{a_{N-1}} \times ... \times \mathbb{Z}_N^{a_1} \times \mathbb{Z}_N^{a_0}$ be the image of y under this map. Note that for every $0 \leq j \leq N-1$, y_j is an a_j-tuple consisting of only 0's and 1's. Let m_j be the number of 1's in y_j (hence $\sum m_j = m$), and let σ_j be a permutation in the symmetric group S_{a_j} so that

$$\sigma_j y_j = (1^{m_j}, 0^{a_j - m_j}) \tag{3.11}$$

Now, let each of the σ_j act on $p \in \mathbb{Z}_N^k$ by

$$\bar{\sigma}_j p = \alpha^{-1}(p_{N-1}, ..., \sigma_j p_j, ..., p_1, p_0),$$

where $(p_{N-1}, ..., p_1, p_0) = \alpha(p)$. and set $\sigma = \bar{\sigma}_0 \ldots \bar{\sigma}_{N-1}$. Let $(a_{N-1}, ..., a_1, a_0) = \alpha(\hat{a})$ so equation (3.9) gives $a_j = (j^{a_j})$ and σ stabilizes $\hat{a}$, that is,

$$\sigma\hat{a} = \hat{a}. \tag{3.12}$$

From (3.11) and the definition of σ, we get that:

$$\begin{aligned} \sigma y &= \sigma(y'_{N-1}, ..., y'_1, y'_0) \\ &= (1^{m_{N-1}}, 0^{a_{N-1}-m_{N-1}}, ..., 1^{m_1}, 0^{a_1-m_1}, 1^{m_0}, 0^{a_0-m_0}). \end{aligned} \tag{3.13}$$

We can write $\hat{a}$ as follows:

$$\begin{aligned} \hat{a} &= ((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0^{a_0}) \\ &= \left((N-1)^{m_{N-1}}, (N-1)^{a_{N-1}-m_{N-1}}, ..., 1^{m_1}, 1^{a_1-m_1}, 0^{m_0}, 0^{a_0-m_0}\right). \end{aligned} \tag{3.14}$$

Now since the action of S_k on $\mathbb{Z}_N^k$ is linear, we get from equations (3.10), (3.12), (3.13) and (3.14) that:

$$\begin{aligned} \sigma z &= \hat{a} + \sigma y \\ &= \left(N^{m_{N-1}}, (N-1)^{a_{N-1}-m_{N-1}}, ..., 2^{m_1}, 1^{a_1-m_1}, 1^{m_0}, 0^{a_0-m_0}\right) \\ &= \left(N^{m_{N-1}}, (N-1)^{a_{N-1}-m_{N-1}+m_{N-2}}, ..., 1^{a_1-m_1+m_0}, 0^{a_0-m_0}\right) \\ &= \left(0^{m_{N-1}}, (N-1)^{a_{N-1}-m_{N-1}+m_{N-2}}, ..., 1^{a_1-m_1+m_0}, 0^{a_0-m_0}\right) \end{aligned} \tag{3.15}$$

which finishes the proof of the claim.

Now, to complete the proof of the theorem we see that from Remark 3.0.4 we have that

$$M^{(k)[c]}_{[a],[b]} = \text{ the number of non-redundant equations of the form (3.10) },$$

where $y \in [b]$ and $z \in [c]$.

Since $M^{(k)[c]}_{[a],[b]} \neq 0$, there exists $y \in [b]$ and $z \in [c]$ satisfying (3.10). We have to prove that if $y' \in [b]$ and $z' \in [c]$ also satisfy (3.10), then there exists $\gamma \in S_k$ so that

$$\gamma\hat{a} = \hat{a}, \qquad \gamma y = y' \qquad \text{and} \qquad \gamma z = z'.$$

From the claim we have that since $z = \hat{a} + y$, then there exits $\sigma \in S_k$ such that $\sigma\hat{a} = \hat{a}$,

$$\sigma y = (1^{m_{N-1}}, 0^{a_{N-1}-m_{N-1}}, ..., 1^{m_1}, 0^{a_1-m_1}, 1^{m_0}, 0^{a_0-m_0}) \tag{3.16}$$

and

$$\sigma z = \left(0^{m_{N-1}}, (N-1)^{a_{N-1}-m_{N-1}+m_{N-2}}, ..., 1^{a_1-m_1+m_0}, 0^{a_0-m_0}\right), \tag{3.17}$$

for some integers m_0, m_1,...,m_{N-1} such that $0 \leq m_i \leq a_i$, for $0 \leq i \leq N-1$ and $m_0 + m_1 + ... + m_{N-1} = m$.

From Equation (3.17) it follows that the representative in standard form, $\hat{c}$, from the orbit $[c] = [\sigma z] = [z]$ has the form

$$\hat{c} = \left((N-1)^{a_{N-1}-m_{N-1}+m_{N-2}}, ..., 1^{a_1-m_1+m_0}, 0^{a_0-m_0+m_{N-1}}\right). \tag{3.18}$$

Now, since $z' = \hat{a} + y'$, the claim also guarantees the existence of a permutation $\sigma' \in S_k$ so that $\sigma'\hat{a} = \hat{a}$,

$$\sigma' y' = (1^{t_{N-1}}, 0^{a_{N-1}-t_{N-1}}, ..., 1^{t_1}, 0^{a_1-t_1}, 1^{t_0}, 0^{a_0-t_0}) \tag{3.19}$$

and

$$\sigma' z' = \left(0^{t_{N-1}}, (N-1)^{a_{N-1}-t_{N-1}+t_{N-2}}, ..., 1^{a_1-t_1+t_0}, 0^{a_0-t_0}\right) \tag{3.20}$$

for some integers t_0, t_1,...,t_{N-1} such that $0 \leq t_i \leq a_i$, for $0 \leq i \leq N-1$ and $t_0 + t_1 + ... + t_{N-1} = m$. From Equation (3.20) it follows that

$$\hat{c} = \left((N-1)^{a_{N-1}-t_{N-1}+t_{N-2}}, ..., 1^{a_1-t_1+t_0}, 0^{a_0-t_0+t_{N-1}}\right). \tag{3.21}$$

Since the representative in standard form $\hat{c}$ of the orbit $[c]$ is unique, from Equations (3.18) and (3.21), we get the system of equations

$$\begin{gathered}
a_{N-1} - t_{N-1} + t_{N-2} = a_{N-1} - m_{N-1} + m_{N-2} \\
a_{N-2} - t_{N-2} + t_{N-3} = a_{N-2} - m_{N-2} + m_{N-3} \\
\vdots \\
a_1 - t_1 + t_0 = a_1 - m_1 + m_0 \\
a_0 - t_0 + t_{N-1} = a_0 - m_0 + m_{N-1} \\
t_0 + t_1 + ... + t_{N-1} = m = m_0 + m_1 + ... + m_{N-1}.
\end{gathered}$$

Which is equivalent to the system

$$
\begin{gathered}
-t_{N-1}+t_{N-2}=-m_{N-1}+m_{N-2} \\
-t_{N-2}+t_{N-3}=-m_{N-2}+m_{N-3} \\
\vdots \\
-t_1+t_0=-m_1+m_0 \\
-t_0+t_{N-1}=-m_0+m_{N-1} \\
t_0+t_1+...+t_{N-1}=m=m_0+m_1+...+m_{N-1},
\end{gathered}
$$

and it can be easily seen that this system has as unique solution $t_i = m_i$ for $i = 0, 1, ..., N-1$.

Therefore we have that $\sigma\hat{a} = \hat{a} = \sigma'\hat{a}$, from Equations (3.16) and (3.19) we have that $\sigma y = \hat{a} = \sigma' y'$ and from Equations (3.17) and (3.20) we have that $\sigma z = \hat{a} = \sigma' z'$. Hence by letting $\gamma = \sigma^{-1}\sigma'$ we get

$$\gamma\hat{a} = \hat{a} \qquad \gamma y' = y \qquad \text{and} \qquad \gamma z' = z.$$

Thus we have $M^{(k)[c]}_{[a],[b]} = 1$. □

Corollary 3.0.10. *Let $[a] \in \mathcal{O}$, assume that $\hat{a} = ((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0^{a_0})$ and let $[b] = [(1^m, 0^{k-m})]$. Then for $[c] \in \mathcal{O}$ we have*

$$
M^{(k)[c]}_{[a],[b]} = \begin{cases} 1, & \text{if } \hat{c} = ((N-1)^{a_{N-1}-m_{N-1}+m_{N-2}}, ..., 1^{a_1-m_1+m_0}, 0^{a_0-m_0+m_{N-1}}), \\ & \text{for some integers } m_0,\ m_1,...,m_{N-1} \text{ such that } \displaystyle\sum_{i=0}^{N-1} m_i = m \\ & \text{and} \quad 0 \le m_i \le a_i \quad , \text{for} \quad 0 \le i \le N-1 \\ 0, & \text{otherwise.} \end{cases}
$$

We finish the chapter with an example showing that the coefficient $M^{(k)[c]}_{[a],[b]}$ could be more than 1.

Example 3.0.11. *Let $a = (2,1,0) \in \mathbb{Z}_3^3$. Then we have that $\hat{a} = (2,1,0)$ and $[a] = \{(2,1,0), (2,0,1), (0,2,1), (0,1,2), (1,0,2), (1,2,0)\}$. To compute $[a] \times [a]$, we* *remove all redundancies from the list:*

$$(2,1,0) + (2,1,0) = (1,2,0),$$
$$(2,1,0) + (2,0,1) = (1,1,1),$$
$$(2,1,0) + (0,2,1) = (2,0,1),$$
$$(2,1,0) + (0,1,2) = (2,2,2),$$
$$(2,1,0) + (1,0,2) = (0,1,2),$$
$$(2,1,0) + (1,2,0) = (0,0,0).$$

We can see that there are no redundant equations in that list, so we get

$$[(2,1,0)] \times [(2,1,0)] = [(2,2,2)] + [(1,1,1)] + 3[(2,1,0)] + [(0,0,0)]. \tag{3.22}$$

Note that the orbit $[(2,1,0)]$ *has multiplicity 3 in the product.*

Chapter 4

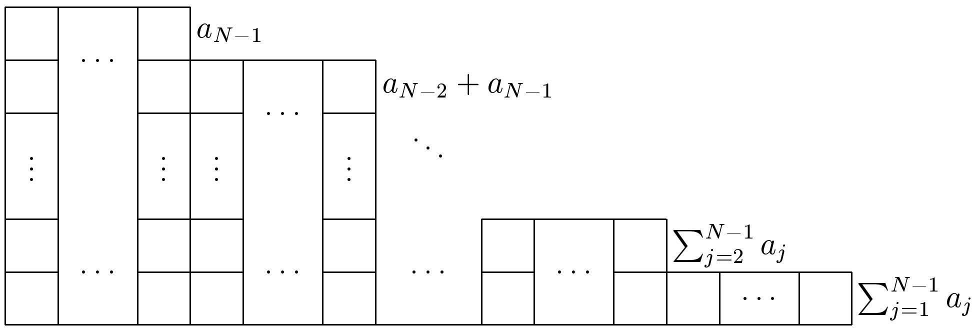
Young diagrams and Pieri rules

4.1 Symmetric polynomials and fusion algebras

Given a dominant integral weight of $\mathfrak{g} = sl_N$, $\mu = \sum_{j=1}^{N-1} a_j \lambda_j$, we have associated to it a partition, denoted by (μ), given in (2.11) by:

$$(\mu) = \left(\sum_{j=1}^{N-1} a_j, \sum_{j=2}^{N-1} a_j, ..., a_{N-1} \right) = (\mu_1, \mu_2, ..., \mu_{N-1}) \tag{4.1}$$

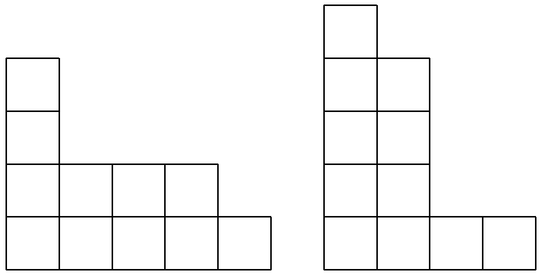
where μ_t, $1 \le t \le N-1$, is the t^{th} part of the partition (μ). To such partition we can associate a Young diagram which is defined as the set of unit squares centered at the points $(s,t) \in \mathbb{Z}^2$ for $1 \le s \le \mu_t$ and $1 \le t \le l(\mu)$, where $l(\mu)$ denotes the length of (μ), the largest value of t such that $\mu_t \neq 0$. The Young diagram associated to the partition (4.1) is given below.



where the label at the end of every row means the length of the row.

Definition 4.1.1. *Given a Young diagram associated to a partition* $(\mu) = (\mu_1, \mu_2, ..., \mu_N)$*, we define the* ***conjugate Young diagram*** *to be the diagram obtained from the Young diagram of* (μ) *by interchanging rows and columns. The partition associated to the conjugate Young diagram will be denoted by* $(\tilde{\mu}) = (\tilde{\mu}_{\mathbf{1}}, \tilde{\mu}_{\mathbf{2}}, ..., \tilde{\mu}_{\mathbf{t}})$ *where* $t = \mu_1$ *and will be called* ***the conjugate of*** (μ).

Example 4.1.2. *Let* $(\mu) = (5, 4, 1, 1)$*. Below we have the diagrams of* (μ) *and its conjugate* $(\tilde{\mu})$.



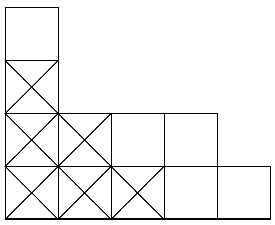
(μ) $\qquad$ $(\tilde{\mu})$

Then the conjugate of (μ) *is the partition* $(\tilde{\mu}) = (4, 2, 2, 2, 1)$*. Note that the length of* $(\tilde{\mu})$ *equals* μ_1.

The following definitions are important to describe the product of symmetric polynomials.

Definition 4.1.3. *If* (ν) *and* (μ) *are partitions so that* $\mu_i \leq \nu_i$ *for all* i*, the set difference between* (ν) *and* (μ) *is called a* ***skew partition*** *denoted by* $(\nu)/(\mu)$ *and its diagram is called a* ***skew diagram***.

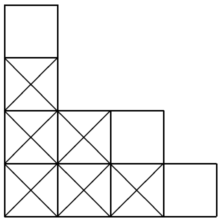
Example 4.1.4. *Let* $(\mu) = (3, 2, 1, 0)$ *and* $(\nu) = (5, 4, 1, 1)$*, the Young diagrams for* (ν)*,* (μ) *and* $(\nu)/(\mu)$ *are below*



where (ν) is the whole diagram, (μ) is the diagram formed by the crossed boxes and the skew diagram $(\nu)/(\mu)$ is formed by the empty boxes.

Definition 4.1.5. *A skew diagram is called an* ***m-column strip*** *if it has m boxes with at most one box in each row, and is called an* ***m-row strip*** *if it has m boxes with at most one box in each column.*

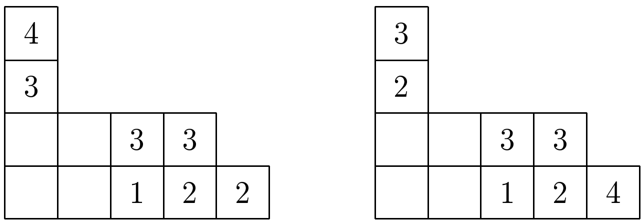
Example 4.1.6. *Let* $(\mu) = (3, 2, 1, 0)$ *and* $(\nu) = (4, 3, 1, 1)$



The skew diagram $(\nu)/(\mu)$ in this example is both a 3-row strip and a 3-column strip.

Definition 4.1.7. *A* ***tableau*** *of shape $(\nu)/(\mu)$ is a filling of the diagram $(\nu)/(\mu)$ with positive integers nondecreasing in rows and strictly increasing in columns. The* ***content*** *of a tableau is the sequence $(b_1, b_2, ...)$ where i appears b_i times in the filling for every $i \geq 1$.*

Example 4.1.8. *Let* $(\mu) = (2, 2)$ *and* $(\nu) = (5, 4, 1, 1)$*. The following fillings of the skew diagram $(\nu)/(\mu)$ give two tableaux*



These two tableaux have both content (1,2,3,1).

Young diagrams are closely related to the algebra of symmetric polynomials, since a very important basis for this algebra is indexed by partitions. We describe the algebra of symmetric polynomials as well as its basis formed by Schur polynomials in the following paragraph.

The algebra of symmetric polynomials in N variables is the algebra of polynomials $f \in \mathbb{Q}[x_1, x_2, ..., x_N]$ invariant under the action of the symmetric group S_N that permutes the variables. This algebra is denoted by $\mathbf{\Lambda_N} = \mathbb{Q}[\mathbf{x_1}, \mathbf{x_2}, ..., \mathbf{x_N}]^{\mathbf{S_N}}$.

For $0 < m \leq N$ define the elementary symmetric polynomial

$$e_m = \sum_{1 \leq i_1 < ... < i_m \leq N} x_{i_1} ... x_{i_m},$$

and $e_m = 0$ for $m > N$.

For $m > 0$ we define the homogeneous symmetric function

$$h_m = \sum_{1 \leq i_1 ... \leq i_m \leq N} x_{i_1} ... x_{i_m} \ ,$$

and we define $e_m = 0 = h_m$ for $m < 0$. We also define $e_0 = 1 = h_0$.

A basis for Λ_N is given by the Schur polynomials $S_{(\mu)}$ indexed by partitions $(\mu) = (\mu_1, \mu_2, ..., \mu_N)$ with $l(\mu) \leq N$, defined by

$$S_{(\mu)} = det(h_{\mu_i - i + j}) = \begin{vmatrix} h_{\mu_1} & h_{\mu_1+1} & \cdots & h_{\mu_1+N-1} \\ h_{\mu_2-1} & h_{\mu_2} & \cdots & h_{\mu_2+N-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\mu_N-N+1} & h_{\mu_N-N+2} & \cdots & h_{\mu_N} \end{vmatrix} \tag{4.2}$$

if $1 \leq l(\mu) \leq N$. Note that $S_{(\mu)} = h_m$ if $(\mu) = (m)$ consists of a single part.

The elementary symmetric polynomials are also generators of the algebra of Λ_N, since Schur polynomials can also be expressed as a determinant of them. If $(\mu) = (\mu_1, \mu_2, ..., \mu_N)$ and its conjugate partition is $(\tilde{\mu}) = (\tilde{\mu}_1, \tilde{\mu}_2, ..., \tilde{\mu}_t)$ where $t = \mu_1$, then the Schur polynomial $S_{(\mu)}$ is also given by

$$S_{(\mu)} = det(e_{\tilde{\mu}_i - i + j}) = \begin{vmatrix} e_{\tilde{\mu}_1} & e_{\tilde{\mu}_1+1} & \cdots & e_{\tilde{\mu}_1+t-1} \\ e_{\tilde{\mu}_2-1} & e_{\tilde{\mu}_2} & \cdots & e_{\tilde{\mu}_2+t-2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\tilde{\mu}_t-t+1} & e_{\tilde{\mu}_t-t+2} & \cdots & e_{\tilde{\mu}_t} \end{vmatrix}. \tag{4.3}$$

Note that $S_{(\mu)} = e_m$ if $(\mu) = (1^m)$.

Equations (4.2) and (4.3) are known in the literature as the Jacobi-Trudi determinants.

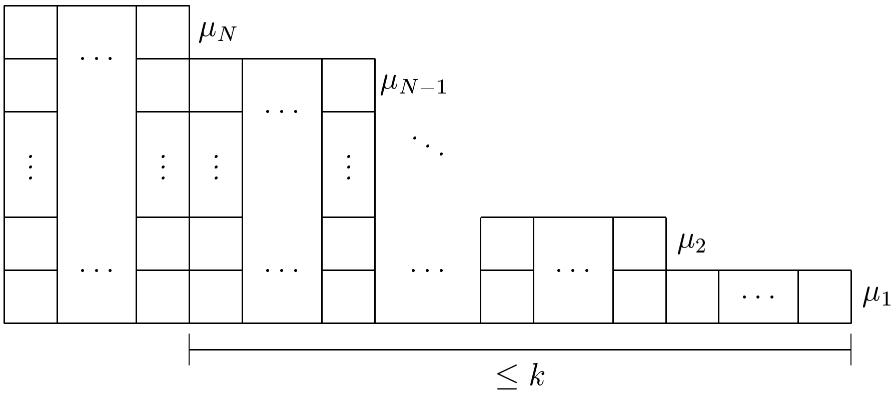
It was proved by Goodman and Wenzl [GW] that the fusion algebra $\mathcal{F}(A_{N-1}, k)$ associated to $\widehat{sl}_N$ on level k is isomorphic to the quotient algebra of symmetric polynomials $\Lambda_N/I^{(N,k)}$, where

$$I^{(N,k)} = \langle S_{(1^N)} - 1, S_{(\mu)} \mid \mu_1 - \mu_N = k+1 \rangle.$$

This condition on partitions in $I^{(N,k)}$ limits the shape of Young diagrams that are relevant to the quotient algebra $\Lambda_N/I^{(N,k)}$. The following definition exactly describes these diagrams.

Definition 4.1.9. *Let (μ) be a partition so that $\mu_1 - \mu_N \leq k$, then we say that (μ) is $(\mathbf{N}, \mathbf{k})$-**restricted**. The set of (N,k)-restricted partitions will be denoted by $\mathbf{\Pi}^{(\mathbf{N},\mathbf{k})}$.*

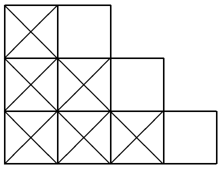
Example 4.1.10. *The Young diagrams for the partitions in the definition are exactly the ones whose distance between the first column of height $N-1$ and the last column, is less than or equal to k, as the next diagram shows.*



After giving some definitions we present below some results of Goodman and Wenzl, (see [GW] or [Tu]) which provide the multiplication in the fusion algebra $\mathcal{F}(A_{N-1}, k)$.

Definition 4.1.11. *We call a skew diagram $(\nu)/(\mu)$ a $(\mathbf{N}, \mathbf{k})$-**cylindric m-row strip** if it is an m-row strip for some m and $\nu_1 - \mu_N \leq k$.*

Example 4.1.12. *Let $\mu = (3,2,1)$ and $\nu = (4,3,2)$ be partitions, $N = 3$ and $k = 3$. The skew partition ν/μ is a (3,3)-cylindric 3-row strip, since $\nu_1 - \mu_3 = 3$. This can be visualized from the diagram*



In fact ν/μ is a $(3,k)$-cylindric 3-row strip for any $k \geq 3$ and it is not a $(3,2)$-cylindric 3-row strip.

The isomorphism $\Lambda_N / I^{(N,k)} \cong \mathcal{F}(A_{N-1}, k)$ gives the following dictionary

$$\begin{aligned} \Lambda_N^{S_N}/I^{(N,k)} &\longleftrightarrow \mathcal{F}(A_{N-1},k) \\ S_{(\mu)} &\longleftrightarrow V^{\mu} \\ S_{(\mu)}S_{(\lambda)} = \sum_{(\nu)} N^{(k)(\nu)}_{(\mu),(\lambda)} S_{(\nu)} &\longleftrightarrow V^{\mu} \otimes_k V^{\lambda} = \bigoplus_{\nu} N^{(k)\nu}_{\mu,\lambda} V^{\nu}, \end{aligned} \tag{4.4}$$

where the left hand side is indexed by partitions $(\mu) = (\mu_1, ..., \mu_N) \in \Pi^{(N,k)}$ and the right hand side is indexed by weights $\mu = \sum_{j=1}^{N-1}(\mu_j - \mu_{j+1})\lambda_j \in P_k^+$. The correspondence (2.10) allows us to move from left to right in the dictionary and the right inverse (2.11) of (2.10) allows us to move in the opposite direction in the dictionary. We also get an equality of structure constants in both rings, i.e., $N^{(k)(\nu)}_{(\mu),(\lambda)} = N^{(k)\nu}_{\mu,\lambda}$.

4.2 Fusion Pieri rules

From the equivalence relation $\sim$ defined in (2.12) we know that given a weight $\mu = \sum_{j=1}^{N-1} a_j \lambda_j \in P_k^+$ there are infinitely many partitions in $\Pi^{(N,k)}$ that correspond to μ, since the correspondence (2.11) is onto but not one-to-one. The next lemma says that the Schur polynomials corresponding to partitions that are equivalent under $\sim$

are equal in the quotient $\Lambda_N/I^{(N,k)}$. Just recall that two partitions $(\mu) = (\mu_1, ..., \mu_N)$ and $(\nu) = (\nu_1, ..., \nu_N)$ are equivalent under $\sim$ if and only if there exists a positive integer c such that $\mu_i = \nu_i + c$ for $i = 1, ..., N$. Then it would be sufficient to prove the following.

Lemma 4.2.1. *Let $(\mu) = (\mu_1, ..., \mu_N) \in \Pi^{(N,k)}$, then in $\Lambda_N/I^{(N,k)}$ we have*

$$S_{(\mu)} = S_{(\mu_1-\mu_N,...,\mu_{N-1}-\mu_N)}.$$

To prove this lemma we need the following result due to Goodman-Wenzl (see [GW] or [Tu]).

Theorem 4.2.2. *[GW] (**Fusion Pieri rule** for multiplication by e_m)*

Let $(\mu) \in \Pi^{(N,k)}$ and $m \leq N$. Then in $\Lambda_N/I^{(N,k)}$ we have

$$S_{(\mu)}e_m = \sum_{\substack{(\nu)\in\Pi^{(N,k)}, \\ (\nu)/(\mu) \text{ is an } m\text{-column strip}}} S_{(\nu)}. \tag{4.5}$$

Corollary 4.2.3. *For $(\mu) = (\mu_1, ..., \mu_N) \in \Pi^{(N,k)}$ we have the equality in $\Lambda_N/I^{(N,k)}$*

$$S_{(\mu)} = S_{(\mu_1+1,...,\mu_N+1)}.$$

Proof. From Theorem 4.2.2 we have

$$S_{(\mu)}e_N = \sum_{\substack{(\nu)\in\Pi^{(N,k)}, \\ (\nu)/(\mu) \text{ is an N-column strip}}} S_{(\nu)}.$$

The partition (μ) has length at most N, so its Young diagram has at most N rows, therefore the only $(\nu) \in \Pi^{(N,k)}$ such that $(\nu)/(\mu)$ is an N-column strip is the one obtained by adding one box on each row, so we get

$$S_{(\mu)}e_N = S_{(\mu_1+1,...,\mu_N+1)}.$$

Now, since $e_N - 1$ is in $I^{(N,k)}$, we get the desired equality in $\Lambda_N/I^{(N,k)}$. $\square$

Lemma 4.2.1 is a direct consequence of this corollary. It can be also proved directly using (4.3).

Example 4.2.4. *Let $N = 4$, $k = 3$ and $(\mu) = (5,4,4,3)$. From Lemma 4.2.1 we have that in $\Lambda_4/I^{(4,3)}$*

$$S_{(\mu)} = S_{(2,1,1)}.$$

We can actually see this equality by using the definition of the Schur polynomial in terms of the elementary symmetric polynomials and the fact that $e_4 = 1$ in the quotient algebra $\Lambda_4/I^{(4,3)}$. The conjugate of $(\mu) = (5,4,4,3)$ is the partition $(\tilde{\mu}) = (4,4,4,3,1)$ so from the Jacobi-Trudi determinant given by (4.3) we get

$$S_{(\mu)} = det(e_{\tilde{\mu}_i - i + j}) = \begin{vmatrix} e_4 & e_5 & e_6 & e_7 & e_8 \\ e_3 & e_4 & e_5 & e_6 & e_7 \\ e_2 & e_3 & e_4 & e_5 & e_6 \\ e_0 & e_1 & e_2 & e_3 & e_4 \\ e_{-3} & e_{-2} & e_{-1} & 1 & e_1 \end{vmatrix}.$$

Using the relations $e_{-1} = e_{-2} = e_{-3} = e_5 = e_6 = e_7 = e_8 = 0$ and $e_0 = 1$ in Λ_4 and $e_4 = 1$ in $\Lambda_4/I^{(4,3)}$, this determinant simplifies to

$$S_{(\mu)} = det(e_{\tilde{\mu}_i - i + j}) = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ e_3 & 1 & 0 & 0 & 0 \\ e_2 & e_3 & 1 & 0 & 0 \\ 1 & e_1 & e_2 & e_3 & 1 \\ 0 & 0 & 0 & 1 & e_1 \end{vmatrix} = \begin{vmatrix} e_3 & 1 \\ 1 & e_1 \end{vmatrix}.$$

The conjugate of the partition $(2,1,1)$ is the partition $(3,1)$, and from (4.3) we get

$$S_{(2,1,1)} = det(e_{\tilde{\mu}_i - i + j}) = \begin{vmatrix} e_3 & 1 \\ 1 & e_1 \end{vmatrix}.$$

This example shows how Lemma 4.2.1 yields the equality in a much simpler way.

Lemma 4.2.1 says that from a Young diagram in $\Pi^{(N,k)}$ we can remove all of the columns of height N, and the Schur polynomials corresponding to the original diagram and the new diagram are equal in $\Lambda_N/I^{(N,k)}$.

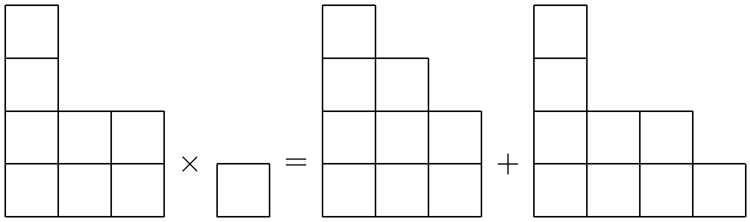
The next theorem illustrates another result of [GW], connected to the orbits product from Theorem 3.0.9. We will see some examples of how the connection works in this chapter and we will give the proof in the next chapter.

Theorem 4.2.5. *[GW] (**Fusion Pieri rule** for multiplication by h_m)*

Let $(\mu) \in \Pi^{(N,k)}$ and $m \leq k$. Then in $\Lambda_N/I^{(N,k)}$ we have

$$S_{(\mu)}h_m = \sum_{\substack{(\nu)\in\Pi^{(N,k)},\\ (\nu)/(\mu) \text{ is an } (N,k)\text{-cylindric } m\text{-row strip}}} S_{(\nu)}. \tag{4.6}$$

Example 4.2.6. *Let $N = 4$, $k = 3$ and $(\mu) = (3,3,1,1)$. Using the previous theorem we get that $S_{(\mu)}h_1$ equals the sum of $S_{(\nu)}$'s so that (ν) is a $(4,3)$-cylindric 1-row strip. The diagram below shows all possible (ν)'s.*



Therefore in the algebra $\Lambda_4/I^{(4,3)} = \mathbb{Q}[x_1,x_2,x_3,x_4]^{S_4}/I^{(4,3)}$ we get that

$$S_{(3,3,1,1)}h_1 = S_{(3,3,2,1)} + S_{(4,3,1,1)}.$$

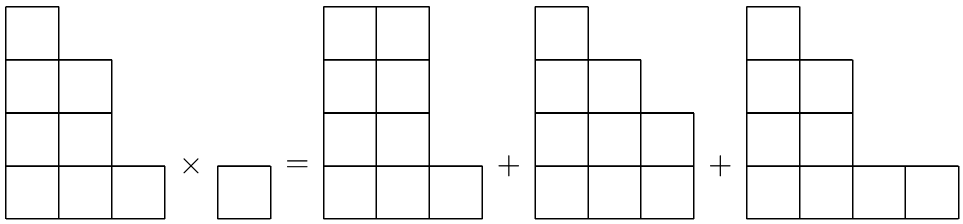
From Lemma 4.2.1 we know that we can remove the column of length 4 from each of the Young diagrams, so we get

$$S_{(2,2)}h_1 = S_{(2,2,1)} + S_{(3,2)}. \tag{4.7}$$

Now using the Dictionary (4.4), this product translates into the fusion product in $\mathcal{F}(A_3, 3)$

$$V^{2\lambda_2} \otimes_3 V^{\lambda_1} = V^{\lambda_2+\lambda_3} \oplus V^{\lambda_1+2\lambda_2}.$$

Example 4.2.7. *Let $N = 4$, $k = 3$ and $(\mu) = (3,2,2,1)$. To find the product $S_{(\mu)}h_1$, we use the diagram below, which shows all diagrams $(\nu) \in \Pi^{(4,3)}$ where $(\nu)/(\mu)$ is a (4,3)-cylindric 1-row strip*



Therefore, in the algebra $\Lambda_4/I^{(4,3)}$, we get that

$$S_{(3,2,2,1)}h_1 = S_{(3,2,2,2)} + S_{(3,3,2,1)} + S_{(4,2,2,1)}.$$

By removing all columns of height 4 in each of the diagrams on both sides we get

$$S_{(2,1,1)}h_1 = S_{(1)} + S_{(2,2,1)} + S_{(3,1,1)} = h_1 + S_{(2,2,1)} + S_{(3,1,1)}, \tag{4.8}$$

which, by the Dictionary (4.4), translates into the fusion product in $\mathcal{F}(A_3, 3)$

$$V^{\lambda_1+\lambda_3} \otimes_3 V^{\lambda_1} = V^{\lambda_1} \oplus V^{\lambda_2+\lambda_3} \oplus V^{2\lambda_1+\lambda_3}$$

The previous theorem can be simplified. Note that since μ is a weight of A_{N-1}, we can take the Young diagram for (μ) to be inside the rectangle $(N-1)\times k$, without loss of generality. Then (ν) in the right hand side of (4.6) has to be inside the rectangle $N\times k$. We prove this statement in the following lemma after setting up some notation.

Notation 4.2.8. *For a partition $(\mu) \in \Pi^{(N,k)}$, we write $(\mu) \subseteq r \times k$ when the Young diagram of (μ) is contained in the rectangle $r \times k$.*

Lemma 4.2.9. *Let $(\mu) \subseteq (N-1) \times k$ and let $(\nu) \in \Pi^{(N,k)}$, where $(\nu)/(\mu)$ is a (N,k)-cylindric m-row strip for $m \leq k$, then $(\nu) \subseteq N \times k$.*

Proof. Let $(\mu) \subseteq (N-1) \times k$ and assume that $(\nu) \in \Pi^{(N,k)}$ where $(\nu)/(\mu)$ is a (N,k)-cylindric m-row strip. Thus (μ) is a partition of length at most $N-1$, say $(\mu) = (\mu_1, ..., \mu_{N-1})$. Note that since $(\mu) \subseteq (N-1) \times k$ and $(\nu)/(\mu)$ is an m-row strip, then the height of any column of (ν) is at most N. That is, (ν) is a partition of length at most N, say $(\nu) = (\nu_1, ..., \nu_N)$.

We claim that (ν) cannot have rows of length $> k$, because otherwise $\nu_1 - \mu_N = \nu_1 > k$ which means $(\nu)/(\mu)$ is not (N,k)-cylindric. Therefore (ν) is inside the rectangle $N \times k$.

□

Remark 4.2.10. *The condition $(\nu)/(\mu)$ is (N,k)-cylindric is implied from the assumption that $(\nu) \subseteq N \times k$. This is clear because $\nu_1 \leq k$ implies $\nu_1 - \mu_N \leq k$.*

Lemma 4.2.9 and Remark 4.2.10 allow us to rewrite Theorem 4.2.5 and equation (4.6) as follows:

Theorem 4.2.11. *(**Fusion Pieri rule** for multiplication by h_m)*

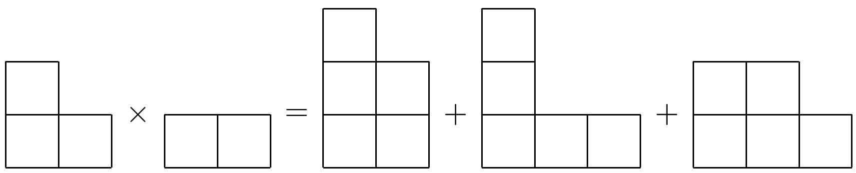
Let $(\mu) \subseteq (N-1) \times k$ and let $m \leq k$. Then in $\Lambda_N / I^{(N,k)}$ we have

$$S_{(\mu)} h_m = \sum_{\substack{(\nu) \subseteq N \times k, \\ (\nu)/(\mu) \text{ is an } m\text{-row strip}}} S_{(\nu)}. \tag{4.9}$$

Remark 4.2.12. *Theorem 4.2.5 seems a little more general than this last theorem, since it applies to partitions (μ) in $\Pi^{(N,k)}$, but in general if $(\mu) = (\mu_1, \mu_2, ..., \mu_N) \in \Pi^{(N,k)}$ we can apply Theorem 4.2.11 to the partition $(\mu_1 - \mu_N, \mu_2 - \mu_N, ..., \mu_{N-1} - \mu_N) \subseteq (N-1) \times k$ and we get the same result.*

Theorem 4.2.11 will be useful to prove our main result (Theorem 5.1.4). Here we show some examples of how this theorem applies.

Example 4.2.13. *Let $(\mu) = (2,1)$, $N = 3$ and $k = 3$. Using Theorem 4.2.11 to compute $S_{(\mu)} h_2$ we get*



that is, in the algebra $\Lambda_3 / I^{(3,3)}$, we have that:

$$S_{(2,1)} h_2 = S_{(2,2,1)} + S_{(3,1,1)} + S_{(3,2)}.$$

Now, the weight associated to the partition $(\mu) = (2,1)$ is $\mu = \lambda_1 + \lambda_2$ and the weight associated to $h_2 = S_{(2,0,0)}$ is $2\lambda_1$. Then the above equation translates into the fusion product in $\mathcal{F}(A_2, 3)$

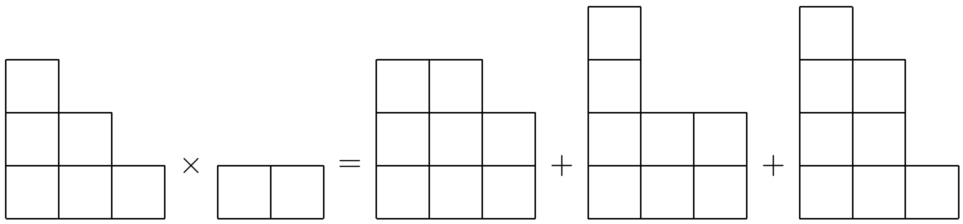
$$V^{\lambda_1 + \lambda_2} \otimes_3 V^{2\lambda_1} = V^{\lambda_2} \oplus V^{2\lambda_1} \oplus V^{\lambda_1 + 2\lambda_2}.$$

Under the correspondence (2.9) we have that the orbits corresponding to the weights $\lambda_1 + \lambda_2$, $2\lambda_1$, λ_2 and $\lambda_1 + 2\lambda_2$ are respectively [(2,1,0)], [(1,1,0)], [(2,0,0)] and [(2,2,1)].

Looking at (3.7) *in Example 3.0.5 we see that this fusion product agrees with the orbit calculation from that example. This is not a coincidence, it follows from* (3.4) *proved by Feingold and Weiner in [FW].*

In the next example we see another fusion product done via Pieri rules whose outcome agrees with an orbit calculation for $N = 4$. This does not follow from (3.4), but it is not a coincidence either as we will see in the next chapter.

Example 4.2.14. *Let* $(\mu) = (3, 2, 1)$*,* $N = 4$ *and* $k = 3$*. Using the Theorem 4.2.11 we get*



This shows that $S_{(3,2,1)}h_2 = S_{(3,3,2)} + S_{(3,3,1,1)} + S_{(3,2,2,1)}$ *where the computation is done in the algebra* $\Lambda_4/I^{(4,3)}$*. By taking away columns of length 4 we get*

$$S_{(3,2,1)}h_2 = S_{(3,3,2)} + S_{(2,2)} + S_{(2,1,1)}.$$

By the Dictionary (4.4) *we get the fusion product*

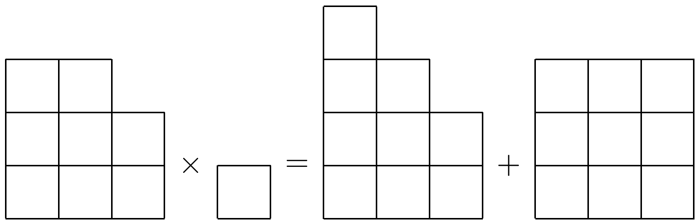
$$V^{\lambda_1+\lambda_2+\lambda_3} \otimes_3 V^{2\lambda_1} = V^{\lambda_2+2\lambda_3} \oplus V^{2\lambda_2} \oplus V^{\lambda_1+\lambda_3}.$$

Under the correspondence (2.9) *we have that the orbits corresponding to the weights* $\lambda_1+\lambda_2+\lambda_3$, $2\lambda_1$, $\lambda_2+2\lambda_3$, $2\lambda_2$ *and* $\lambda_1+\lambda_3$ *are respectively [(3,2,1)], [(1,1,0)], [(3,3,2)], [(2,2,0)] and [(3,1,0)]. Looking at* (3.8) *from Example 3.0.7, we see again that the orbit computation gives exactly the same answer as the Pieri rule computation above.*

Example 4.2.15. *Let* $(\mu) = (3, 2, 1)$*,* $N = 4$ *and* $k = 3$*. Let us compute the product* $S_{(3,2,1)}h_2h_1$ *by iteration. In the previous example we found that* $S_{(3,2,1)}h_2 = S_{(3,3,2)} + S_{(2,2)} + S_{(2,1,1)}$*, therefore we have*

$$S_{(3,2,1)}h_2h_1 = S_{(3,3,2)}h_1 + S_{(2,2)}h_1 + S_{(2,1,1)}h_1.$$

The products $S_{(2,2)}h_1$ and $S_{(2,1,1)}h_1$ were already computed in (4.7) and (4.8) from Examples 4.2.6 and 4.2.7, so we only need to compute the product $S_{(3,3,2)}h_1$. The diagram below shows all diagrams $(\nu) \subseteq N \times k$ such that $(\nu)/(\mu)$ is a 1-row strip, giving the answer according to Theorem 4.2.11



Therefore $S_{(2,1,1)}h_1 = S_{(3,3,2,1)} + S_{(3,3,3)} = S_{(2,2,1)} + S_{(3,3,3)}$. Now putting this together with (4.7) and (4.8), we get

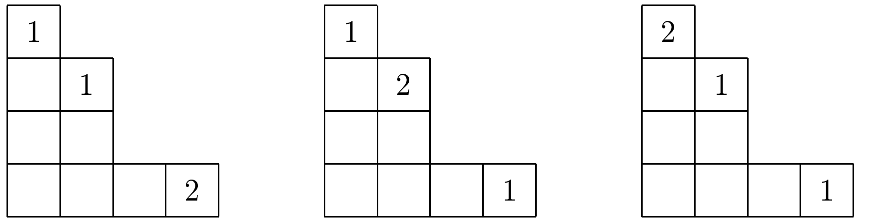
$$S_{(3,2,1)}h_2h_1 = 3S_{(2,2,1)} + S_{(3,3,3)} + S_{(3,2)} + S_{(3,1,1)} + h_1. \tag{4.10}$$

4.3 Iterative fusion Pieri rules

The next theorem illustrates how the iterative product of S_μ by several h_m's, as in the previous example, can be done without iteration if instead of using diagrams we use tableaux. We need a definition before stating the theorem.

Definition 4.3.1. *Let $(\mu) \subseteq (N-1) \times k$, $(\nu) \in \Pi^{(N,k)}$ and $\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_r)$ be a sequence of nonnegative integers. A tableau of shape $(\nu)/(\mu)$ is called an $(\mathbf{N}, \mathbf{k})$-**cylindric tableau** if for each $1 \leq p \leq \nu_N$ the entry in the top row and column p of $(\nu)/(\mu)$ is strictly less than the entry in the bottom row and column $k+p$. We denote by $K^{(N,k)}_{(\nu)/(\mu),\epsilon}$ the number of (N,k)-cylindric tableaux of shape $(\nu)/(\mu)$ and content ϵ. The number $K^{(N,k)}_{(\nu)/(\mu),\epsilon}$ is called the $(\mathbf{N}, \mathbf{k})$-**fusion skew Kostka number** and for $(\mu) = 0$, the number $K^{(N,k)}_{(\nu),\epsilon}$ is called the $(\mathbf{N}, \mathbf{k})$-**fusion Kostka number**.*

Example 4.3.2. *Let $(\mu) = (3,2,1)$, $(\nu) = (4,2,2,1)$, $N = 4$, $k = 3$ and $\epsilon = (2,1)$. Below we draw all tableaux of shape $(\nu)/(\mu)$ and content ϵ and indicate whether each tableau is $(4,3)$-cylindric or not*

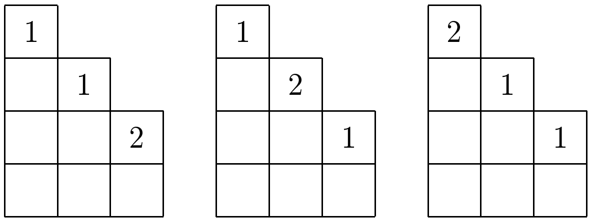


$(3,4)-cylindric$ $\quad$ $not\ (3,4)-cylindric$ $\quad$ $not\ (3,4)-cylindric$

This shows that $K^{(4,3)}_{(4,2,2,1)/(3,2,1),(2,1)} = 1$.

Example 4.3.3. *This example shows the dependence on* k *of the fusion skew Kostka number. Let* (μ), (ν), N *and* ϵ *be as in the previous example and let* $k = 4$. *One can see that all diagrams above are* $(4,4)$*-cylindric, therefore we have that*
$K^{(4,4)}_{(4,2,2,1)/(3,2,1),(2,1)} = 3$.

Example 4.3.4. *Let* $(\mu) = (3,2,1)$, $(\nu) = (3,3,2,1)$, $N = 4$, $k = 3$ *and* $\epsilon = (2,1)$. *Below we show all tableaux of shape* $(\nu)/(\mu)$ *and content* ϵ, *all of them are* $(4,3)$*-cylindric*

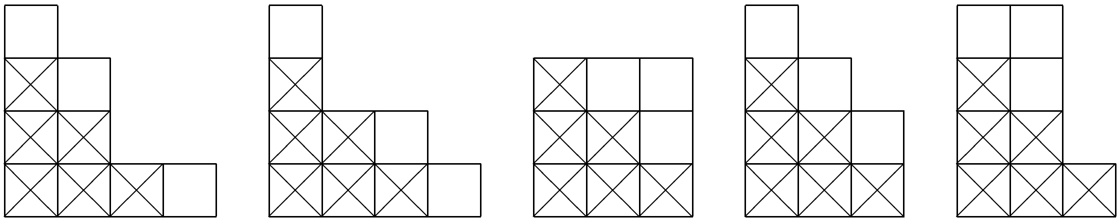


Therefore we have $K^{(4,3)}_{(3,3,2,1)/(3,2,1),(2,1)} = 3$.

Theorem 4.3.5. *Let* $\mu \in P^k_+$, $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_r)$ *a sequence of nonnegative integers and* $h_\epsilon = h_{\epsilon_1} \cdots h_{\epsilon_r}$. *Then we have*

$$S_{(\mu)} h_\epsilon = \sum_{(\nu)\in\Pi^{(N,k)}} K^{(N,k)}_{\nu/\mu,\epsilon} S_{(\nu)}.$$

Example 4.3.6. *Let us use this theorem to compute the product* $S_{(3,2,1)}h_2h_1$, *with* $N = 4$ *and* $k = 3$. *We want to verify that our result matches the answer in Example 4.2.15. According to last theorem, we need to list all diagrams* $(\nu) \in \Pi^{(4,3)}$ *such that* $(\nu)/(\mu)$ *is a* $(\mathbf{4},\mathbf{3})$*-cylindric tableau with content (2,1). Then it is clear that* $(\nu)/(\mu)$ *consists of 3 boxes. We show below all diagrams* $(\nu) \in \Pi^{(4,3)}$ *such that* $(\nu)/(\mu)$ *consists of 3 boxes:*



(1) (2) (3) (4) (5)

Now we have to look at all possible fillings of these skew diagrams that make them into a $(\mathbf{4},\mathbf{3})$*-cylindric tableau. Examples 4.3.2 and 4.3.4 above show the number of* $(4,3)$*-cylindric tableaux for the shapes (1) and (4), and it is easily seen that for the rest of the diagrams the Kostka skew number is equal to 1, so we have:*

$$\begin{aligned} S_{(\mu)}h_2h_1 &= S_{(4,2,2,1)} + S_{(4,3,1,1)} + S_{(3,3,3)} + 3S_{(3,3,2,1)} + S_{(3,2,2,2)} \\ &= S_{(3,1,1)} + S_{(3,2)} + S_{(3,3,3)} + 3S_{(2,2,1)} + S_{(1)} \\ &= S_{(3,1,1)} + S_{(3,2)} + S_{(3,3,3)} + 3S_{(2,2,1)} + h_1. \end{aligned}$$

Just note that this matches (4.10) *from Example 4.2.15.*

4.4 Rank-level duality

Several authors have stated the famous type A rank-level duality, see for instance [Fu1], and many have tried to prove that $\mathcal{F}(A_{N-1},k) \cong \mathcal{F}(A_{k-1},N)$, which is actually not true (for instance $\mathcal{F}(A_1,3)$ is not isomorphic to $\mathcal{F}(A_2,2)$, in fact they do not even have the same dimensions which are 4 and 6, respectively. However see [Tu] or [GW] for a proof). The isomorphism we are going to prove in this section was stated without proof in [Fu1].

Before we start the proof of the isomorphism let's introduce some notation. We have

$$\mathcal{F}(A_{N-1},k) = \Lambda_N/I^{(N,k)} = \mathbb{Q}[x_1,x_2,...,x_N]^{S_N}/I^{(N,k)}$$

is the algebra defined in Section 4.1 and

$$\mathcal{F}(A_{k-1},N) = \Lambda_k/I^{(k,N)} = \mathbb{Q}[y_1,y_2,...,y_k]^{S_k}/I^{(k,N)}.$$

We keep the notation $S_{(\mu)}$ for Schur polynomials in $\Lambda_N/I^{(N,k)}$. To avoid confusion the Schur polynomials in $\Lambda_k/I^{(k,N)}$ will be denoted by $S'_{(\tilde{\mu})}$. We have to use $(\tilde{\mu})$ to index Schur polynomials in Λ_k, because $\Lambda_N/I^{(N,k)}$ is indexed by partitions inside the rectangle $N\times k$ while $\Lambda_k/I^{(k,N)}$ is indexed by partitions inside the rectangle $k\times N$. We also keep the notation $h_m = h_m(x_1, x_2, ..., x_N)$ and $e_s = e_s(x_1, x_2, ..., x_N)$ for h_m and e_s in $\Lambda_N/I^{(N,k)}$, but we will use h'_s to denote $h_s(y_1, y_2, ..., y_k)$ and e'_m for $e_m(y_1, y_2, ..., y_k)$ in $\Lambda_k/I^{(k,N)}$, where $1 \leq s \leq N$ and $1 \leq m \leq k$.

Let $(\mu) \subseteq (N-1) \times k$ be a partition. Equation (4.5) from Theorem 4.2.2 says that in $\Lambda_N/I^{(N,k)}$ we have

$$S_{(\mu)}e_m = \sum_{\substack{(\nu)\in\Pi^{(N,k)}, \\ (\nu)/(\mu) \text{ is an m-column strip}}} S_{(\nu)}.$$

The following remark will be the key in the proof of the rank-level duality isomorphism.

Remark 4.4.1. *If we assume that $\mu_1 < k$, that is, if $(\mu) \subseteq (N-1) \times (k-1)$, then the partitions (ν) on the right hand side of the equation above satisfy $(\nu) \subseteq N \times k$.*

Proof. Assume that $\nu_1 \geq k+1$, then since $\mu_1 \leq k-1$ we have that $\nu_1 - \mu_1 \geq 2$. That means the bottom row of the Young diagram of (ν) has at least two more boxes than the bottom row of the Young diagram of (μ). This contradicts the fact that $(\nu)/(\mu)$ is an m-column strip. □

From this remark we get the following

Lemma 4.4.2. *If $(\mu) \subseteq (N-1) \times (k-1)$ then*

$$S_{(\mu)}e_m = \sum_{\substack{(\nu)\subseteq N\times k, \\ (\nu)/(\mu) \textit{ is an m-column strip}}} S_{(\nu)}. \tag{4.11}$$

Consider the partition $(1^m, 0^{N-m})$. Since $e_m = S_{(1^m)}$ and the conjugate of (1^m) is the partition $(\widetilde{1^m}) = (m, 0, 0, ...)$, applying conjugate to the equation (4.11) we have

$$S'_{(\tilde{\mu})}h'_m = \sum_{\substack{(\tilde{\nu})\subseteq k\times N, \\ (\tilde{\nu})/(\tilde{\mu}) \text{ is an m-row strip}}} S'_{(\tilde{\nu})}. \tag{4.12}$$

This equation is exactly the result obtained in Theorem 4.2.11. This suggests that there might be a an algebra map from $\Lambda_N/I^{(N,k)}$ to $\Lambda_k/I^{(k,N)}$ defined by "conjugation"

$$S_{(\mu)} \longrightarrow S'_{(\tilde{\mu})}.$$

This map fails to be an algebra homomorphism since $e_N = S_{(1^N)} = 1 \in \Lambda_N/I^{(N,k)}$ is mapped to $S'_{\widetilde{(1^N)}} = S'_{(N)} = h'_N$ but $h'_N \neq 1$ in $\Lambda_k/I^{(k,N)}$. But if we modify the ideal $I^{(N,k)}$ by adding the relation $h_k = 1$ and modify the ideal $I^{(k,N)}$ by adding the relation $h'_N = 1$, we will see how the above map becomes an isomorphism between the two new quotient algebras.

Consider the ideal $\bar{I}^{(N,k)}$ of $\Lambda_N = \mathbb{Q}[x_1, x_2, ..., x_N]^{S_N}$ defined by

$$\bar{I}^{(N,k)} = \langle S_{(1^N)} - 1, S_{(\mu)}, h_k - 1 \mid \mu_1 - \mu_N = k+1 \rangle$$

and the ideal $\bar{I}^{(k,N)}$ of $\Lambda_k = \mathbb{Q}[y_1, y_2, ..., y_k]^{S_k}$ defined by

$$\bar{I}^{(k,N)} = \langle S'_{(\tilde{1^k})} - 1, S'_{(\tilde{\mu})}, h'_N - 1 \mid \tilde{\mu}_1 - \tilde{\mu}_k = N+1 \rangle.$$

Then we get a homomorphism

$$\begin{aligned} \Lambda_N/\bar{I}^{(N,k)} &\longmapsto \Lambda_k/\bar{I}^{(k,N)} \\ S_{(\mu)} &\longmapsto S'_{(\tilde{\mu})}. \end{aligned} \tag{4.13}$$

The map sends $h_m = S_{(m)}$ to $S'_{(\tilde{m})} = S'_{(1^m)} = e'_m$ for $0 \leq m \leq k-1$ and $e_m = S_{(1^m)}$ to $S'_{(m)} = h'_m$ for $0 \leq m \leq N-1$. Moreover, by Theorem 4.2.11 and Lemma 4.4.2 we get that the product $S_{(\mu)}h_m$ in $\Lambda_N/\bar{I}^{(N,k)}$ matches the product $S'_{(\tilde{\mu})}e'_m$ in $\Lambda_k/\bar{I}^{(k,N)}$ (See (4.11) and (4.12)). We also get that the product $S_{(\tilde{\mu})}e_m$ in $\Lambda_N/\bar{I}^{(N,k)}$ matches the product $S'_{(\mu)}h'_m$ in $\Lambda_k/\bar{I}^{(k,N)}$. Therefore, the homomorphism above sends generators to generators and the corresponding multiplication by each generator in both rings agrees, therefore we have an isomorphism between both algebras. This isomorphism is known as the Type A rank-level duality.

Using the notation for fusion algebras, the Dictionary (4.4) relates $h_k = S_{(k,0...)}$ with $V^{k\lambda_1}$ in $\mathcal{F}(A_{N-1}, k)$ and $h'_N = S_{(N,0...)}$ with $V^{N\lambda_1}$ in $\mathcal{F}(A_{k-1}, N)$, so we get an isomorphism between the algebras

$$\mathcal{F}'(A_{N-1}, k) \cong \mathcal{F}'(A_{k-1}, N),$$

where

$$\mathcal{F}'(A_{N-1}, k) = \mathcal{F}(A_{N-1}, k)/\langle V^{k\lambda_1} - V^0\rangle$$

and

$$\mathcal{F}'(A_{k-1}, N) = \mathcal{F}(A_{k-1}, N)/\langle V^{N\lambda_1} - V^0\rangle.$$

4.5 Simple currents

In the particular case when $m = k$ of Equation (4.9) in Theorem 4.2.11 we get that for $(\mu) \subseteq (N-1) \times k$

$$S_{(\mu)} h_k = \sum_{\substack{(\nu)\subseteq N\times k, \\ (\nu)/(\mu) \text{ is a } k\text{-row strip}}} S_{(\nu)}.$$

Looking at the right hand side of this equation, we get that there is a unique partition $(\nu) \subseteq N \times k$ so that $(\nu)/(\mu)$ is a k-row strip, namely if $(\mu) = (\mu_1, \mu_2, ..., \mu_{N-1})$ then $(\nu) = (k, \mu_1, \mu_2, ..., \mu_{N-1})$. Therefore the equation above becomes

$$S_{(\mu)} h_k = S_{(k,\mu_1,\mu_2,...,\mu_{N-1})} = S_{(k-\mu_{N-1},\mu_1-\mu_{N-1},\mu_2-\mu_{N-1},...,\mu_{N-2}-\mu_{N-1})}. \tag{4.14}$$

The basis elements with this property are called simple currents. We establish the precise definition of a simple current next.

Definition 4.5.1. *Let $B = \{x_a \mid a \in A\}$ be a basis for a fusion algebra $\mathcal{F}$. We say that $x_a \in B$ is a simple current if for any element $x_b \in B$ the product $x_a x_b = x_c \in B$ for some $c \in A$.*

From equation (4.14) we get that h_k and therefore h_k^t for $t \geq 1$ are simple currents in $\Lambda_N/I^{(N,k)}$. In fact, multiplying $S_{(\mu)}$ iteratively by h_k we get

$$S_{(\mu)} h_k^t = S_{(\mu')} \tag{4.15}$$

where

$$(\mu') = (k - \mu_{N-t} + \mu_{N-(t-1)}, k - \mu_{N-t} + \mu_{N-(t-2)}, ..., k - \mu_{N-t} + \mu_{N-1}, k - \mu_{N-t},$$
$$\mu_1 - \mu_{N-t}, ..., \mu_{N-(t+1)} - \mu_{N-t}). \tag{4.16}$$

In particular, if $(\mu) = (k)$, since $S_{(\mu)} = h_k$ we get $h_k^{t+1} = S_{(k^{(t+1)})}$ and therefore $h_k^N = S_{(k^N)} = 1$ in $\Lambda_N/I^{(N,k)}$, since the partition (k^N) consists of k columns of height N.

It is also known that the set of simple currents of the basis for $\Lambda_N/I^{(N,k)}$ indexed by partitions whose Young diagram are inside the $(N-1) \times k$ rectangle, is the set $\{h_k^t \mid t \geq 0\}$ and that this set actually spans a sub-algebra of $\Lambda_N/I^{(N,k)}$. Moreover, since $h_k^N = 1$ in $\Lambda_N/I^{(N,k)}$, we get that the set of simple currents forms a cyclic group under multiplication isomorphic to $\mathbb{Z}_N$ and the sub-algebra they span is the group algebra $\mathbb{Q}[\mathbb{Z}_N]$.

So we can see that the algebra $\Lambda_N/\bar{I}^{(N,k)}$ can be obtained from the fusion algebra $\Lambda_N/I^{(N,k)}$ by identifying simple currents with 1, in other words by making the quotient by the ideal generated by $h_k - 1$, $\langle h_k - 1\rangle$.

We conclude the chapter by recalling a known characterization of the fusion coefficients $N_{\lambda,\mu}^{(k)\nu}$ when λ is a multiple of the first fundamental weight λ_1 of type A_{N-1}, and an extension of it using the theory of simple currents. The precise characterization is as follows (see [Tu]):

$$N_{\mu,m\lambda_1}^{(k)\nu} = \begin{cases} 1, & \text{if there exists } (\bar{\nu}) \subseteq N \times k \text{ such that } (\bar{\nu})/(\mu) \text{ is an m-row strip} \\ & \text{and } (\bar{\nu}) \sim (\nu), \\ 0, & \text{otherwise,} \end{cases} \tag{4.17}$$

where $m \leq k$ and $\sim$ is the equivalence relation defined in Equation (2.12) and (μ) and (ν) are the partitions corresponding to μ and ν.

Let (μ') be the partition from Equation (4.16) and let (λ) be the partition such that $S_{(\lambda)} = h_k^t h_m$ for $m \leq k$ and $0 \leq t \leq N-1$. Then we get

$$\begin{aligned} S_{(\mu)} S_{(\lambda)} =& S_{(\mu)} h_k^t h_m \\ =& S_{(\mu')} h_m \end{aligned}$$

Therefore we get

$$N_{\mu,\lambda}^{(k)\nu} = N_{\mu',m\lambda_1}^{(k)\nu}. \tag{4.18}$$

Chapter 5

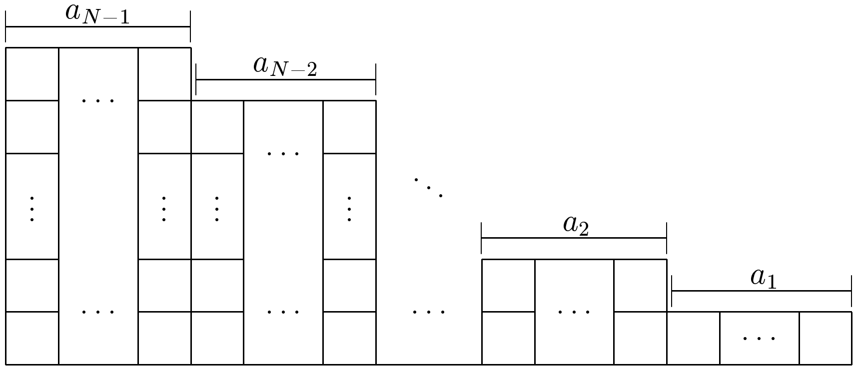
Connection between Young diagrams and S_k-orbits of $\mathbb{Z}_N^k$

5.1 Main theorem and results

Let $\lambda = a_1\lambda_1 + a_2\lambda_2 + ... + a_{N-1}\lambda_{N-1}$ be a dominant integral weight for $\mathfrak{g} = sl_N$ of type A_{N-1}. The orbit in standard form associated to λ is given by (3.2) and the partition is given by (4.1). These two points of view are related as the following lemma shows.

Lemma 5.1.1. *The orbit in standard form* (3.2) *is a partition whose conjugate is given by* (4.1).

Proof. Consider the diagram



where the label a_i on top equals the number of columns in that block, and the subscript i, equals the height of each column in that block. This means that, there are a_{N-1} columns of height $N-1$, a_{N-2} columns of height $N-2$,..., a_2 columns of height 2 and a_1 columns of height 1. Reading the diagram from left to right we get the partition (3.2), and reading the diagram from bottom to top, we get the following: The first row has length $a_{N-1}+a_{N-2}+...+a_1 = \sum_{j=1}^{N-1} a_j$. The second row has length $a_{N-1}+a_{N-2}+...+a_2 = \sum_{j=2}^{N-1} a_j$, and the last row has length a_{N-1}, that is, the diagram above corresponds to the partition (4.1). Thus (3.2) and (4.1) are conjugates of each other. $\square$

Remark 5.1.2. *This lemma says that from a Young diagram $(\mu) \subseteq (N-1)\times k$, we can associate with (μ) a k-tuple whose jth entry is given by the height of the j-th column of the diagram. This k-tuple is actually the representative in standard form of the orbit in $\mathbb{Z}_N^k$ whose corresponding partition is (μ). This k-tuple will be denoted by $\hat{\mu}$.*

Remark 5.1.3. *If $(\nu) = (\nu_1, ..., \nu_N) \subseteq N \times k$ we can also associate with (ν) a k-tuple in $\mathbb{Z}_N^k$, by letting the j-th entry of the k-tuple be the height of the j-th column of (ν) modulo N. This k-tuple is a representative from the orbit $[\nu]$ associated to the partition (ν), but is not necessarily in standard form.* ***We denote this* k*-tuple by*** *$\bar{\nu}$. The representative in standard form for the orbit $[\nu]$ can be obtained directly as follows: Let $(\nu') = (\nu_1 - \nu_N, ..., \nu_{N-1} - \nu_N)$, then $(\nu') \subseteq (N-1)\times k$, so the k-tuple whose j-th entry is the j-th column of the Young diagram for (ν') is the representative in standard form of the orbit $[\nu]$ associated to the partition (ν). Therefore $\nu' \in [\nu]$. We also get that $(\nu) \sim (\bar{\nu}) \sim (\nu')$ where $\sim$ is the equivalence relation defined in* (2.12).

Seeing the standard form of an orbit in $\mathbb{Z}_N^k$ as a partition will help us prove why, in some special cases, the orbit product (3.3) matches the Pieri rules. The next theorem illustrates some of these cases.

Theorem 5.1.4. *Let $\lambda = m\lambda_1$ be a multiple of the first fundamental weight for A_{N-1}, $m \leq k$, $\mu = a_1\lambda_1 + ... + a_{N-1}\lambda_{N-1}$ any other weight of level k and $[\lambda]$ and $[\mu]$ their* *corresponding orbits in $\mathbb{Z}_N^k$. Then $N_{\mu,\lambda}^{(k)\nu} = M_{[\mu],[\lambda]}^{(k)[\nu]}$ for any weight ν of level k. In other words, the product of the orbits $[\mu] \times [\lambda]$ matches Pieri rules for the multiplication $S_{(\mu)}h_m$, (4.9).*

Proof. The first step of the proof is to prove the following claim

Claim: $N_{\mu,\lambda}^{(k)\nu} = 1$ if and only if $M_{[\mu],[\lambda]}^{(k)[\nu]} = 1$

The theorem will follow from this claim, Corollary 3.0.10 and Equation (4.17).

Proof of the claim: Let's assume that $N_{\mu,\lambda}^{(k)\nu} = 1$. From Equation (4.17), we know that there is a partition $(\nu') \subseteq N \times k$ so that $(\nu')/(\mu)$ is an m-row strip, and $(\nu') \sim (\nu)$, then from Remark 5.1.3 we get that $[\nu] = [\nu']$. Since $(\nu') \subseteq N \times k$ and $(\nu')/(\mu)$ is an m-row strip, (ν') can be obtained from (μ) by adding m boxes with no two in the same column, hence there exists $y \in [(1^m, 0^{k-m})]$ so that

$$\hat{\nu} + y = \nu'.$$

Now since $y \in [(1^m, 0^{k-m})]$ and $(\nu') \in [\nu'] = [\nu]$ we have that $M_{[\mu],[\lambda]}^{(k)[\nu]} \neq 0$, then from Corollary 3.0.10 we get $M_{[\mu],[\lambda]}^{(k)[\nu]} = 1$.

In the other direction, let's assume that $M_{[\mu],[\lambda]}^{(k)[\nu]} = 1$. Then from Corollary 3.0.10 we have that $\hat{\nu} = ((N-1)^{a_{N-1}-m_{N-1}+m_{N-2}}, ..., 1^{a_1-m_1+m_0}, 0^{a_0-m_0+m_{N-1}})$, for some integers m_0, m_1,...,m_{N-1} such that $\sum_{i=0}^{N-1} m_i = m$ and $0 \leq m_i \leq a_i$ for $0 \leq i \leq N-1$.

Consider the k-tuple $\bar{\nu} \in \mathbb{Z}_N^k$ defined in Remark 5.1.3,

$$\bar{\nu} = \left(0^{m_{N-1}}, (N-1)^{a_{N-1}-m_{N-1}+m_{N-2}}, ..., 1^{a_1-m_1+m_0}, 0^{a_0-m_0}\right)$$

so $\bar{\nu} \in [\nu]$. Define the Young diagram (ν') whose first m_{N-1} columns have height N and for $j > m_{N-1}$, the j^{th} column has height equal to the j^{th} entry of $\bar{\nu}$.

Now since the k-tuples $\hat{\mu}$ and ν' satisfy the equation

$$\hat{\mu} + (1^{m_{N-1}}, 0^{a_{N-1}+m_{N-1}}, ..., 1^{m_0}, 0^{a_0-m_0}) = \nu' \quad \text{where} \quad \sum_{i=0}^{N-1} m_i = m,$$

then we have that $(\nu')/(\mu)$ is an m-row strip and since $(\nu') \sim (\bar{\nu}) \sim (\nu)$, then we get from (4.17) that $N_{\mu,\lambda}^{(k)\nu} = 1$. □

Note that in general multiplication of two orbits gives a linear combination of orbits. The orbit $[k\lambda_1]$ is special since every product of this orbit with any other gives a single orbit as an answer. We encode this result in the following lemma.

Lemma 5.1.5. *Let $[\mu]$ be any S_k-orbit of $\mathbb{Z}_N^k$. The product $[k\lambda_1] \times [\mu]$ equals a single orbit.*

Proof. Since the k-tuple (1^k) is fixed by the action of S_k, the orbit of $[k\lambda_1] = [(1^k)]$ consists of a single k-tuple. The result easily follows from this fact. □

Although the product defined on orbits is not associative, as shown in example 3.0.8, we can prove associativity in the special case when one of the orbits involved is $[(t^k)]$, $1 \leq t \leq N-1$. We state the result in the following lemma.

Lemma 5.1.6. *Let $[a]$ and $[b]$ be S_k-orbits of $\mathbb{Z}_N^k$, and let $0 \leq t \leq N-1$. Then we have:*

$$\left([a] \times [(t^k)]\right) \times [b] = [a] \times \left([(t^k)] \times [b]\right). \tag{5.1}$$

Proof. From lemma 5.1.5, the products $[a] \times [(t^k)]$ and $[(t^k)] \times [b]$ each consists of a single orbit. If $a = (a_1, ..., a_k)$ is a representative from the orbit $[a]$, then a representative of the orbit $[a] \times [(t^k)]$ is given by $(a_1 + t, ..., a_k + t)$ (where every entry is considered modulo N). Let us denote by $[a+t]$ the orbit of $[a] \times [(t^k)]$, and since the same applies to the orbit $[(t^k)] \times [b]$, we denote this orbit by $[b+t]$, that is,

$$[a] \times [(t^k)] = [a+t] \quad \text{and} \quad [(t^k)] \times [b] = [b+t]. \tag{5.2}$$

Then from (5.1) and (5.2) all we have to do is prove the following equation:

$$[a+t] \times [b] = [a] \times [b+t].$$

This is equivalent to proving that for any orbit $[c]$ of $\mathbb{Z}_N^k$ we have that:

$$M^{(k)[c]}_{[a+t],[b]} = M^{(k)[c]}_{[a],[b+t]}. \tag{5.3}$$

To prove this equality, first observe that there is a one-to-one map given by:

$$\begin{aligned} T([a+t],[b],[c]) &\longrightarrow T([a],[b+t],[c]) \\ (x,y,z) &\longmapsto (x-t, y+t, c) \end{aligned} \tag{5.4}$$

where $x-t$ is the k-tuple obtained from x by subtracting t from every entry of x and $y+t$ is defined by adding t to every entry of y, that is, $x-t = x-(t^k)$ and $y+t = y+(t^k)$. Let $\sigma \in S_k$. Since the action of S_k on k-tuples is linear, we have that $\sigma(x-t) = \sigma(x-(t^k)) = \sigma x - \sigma(t^k)$ and $\sigma(y+t) = \sigma(y+(t^k)) = \sigma y + \sigma(t^k)$. Therefore the map (5.4) is invariant under the action of S_k. Thus the equality (5.3) follows. □

These lemmas are useful to find other orbits whose fusion product can be computed by the orbits method, and those are given below.

Theorem 5.1.7. *For $m \leq k$ let $[m\lambda_1]$ and $[k\lambda_1]$ be the orbits in $\mathbb{Z}_N^k$ corresponding to the weights $m\lambda_1$ and $k\lambda_1$, respectively. For $0 \leq t \leq N-1$ let λ be the weight whose corresponding orbit in $\mathbb{Z}_N^k$ is given by $[\lambda] = [m\lambda_1] \times [k\lambda_1]^t$, and let μ be any other weight on level k. Then for every weight ν on level k we have*

$$N_{\lambda,\mu}^{(k)\nu} = M_{[\lambda],[\mu]}^{(k)[\nu]}.$$

Proof. Since $[k\lambda_1]^t = [(t^k)]$, for an orbit $[\mu]$, we have:

$$[\lambda] \times [\mu] = ([m\lambda_1] \times [k\lambda_1]^t) \times [\mu] = [m\lambda_1] \times ([k\lambda_1]^t \times [\mu]). \tag{5.5}$$

From Lemma 5.1.5 the orbits product $[k\lambda_1]^t \times [\mu] = [\mu + t]$ gives a single orbit. Let μ' be the weight so that $[\mu'] = [\mu + t]$. Then from (5.5) we get

$$[\lambda] \times [\mu] = [m\lambda_1] \times [\mu'].$$

Therefore $M_{[\lambda],[\mu]}^{(k)[\nu]} = M_{[m\lambda_1],[\mu']}^{(k)[\nu]}$, for all weights ν.

From Theorem 5.1.4, we have $M_{[m\lambda_1],[\mu']}^{(k)[\nu]} = N_{m\lambda_1,\mu'}^{(k)\nu}$ for all weights ν. This combined with (4.18) gives us

$$N_{\lambda,\mu}^{(k)\nu} = M_{[m\lambda_1],[\mu']}^{(k)[\nu]}.$$

Therefore we get

$$N_{\lambda,\mu}^{(k)\nu} = M_{[\lambda],[\mu]}^{(k)[\nu]}.$$

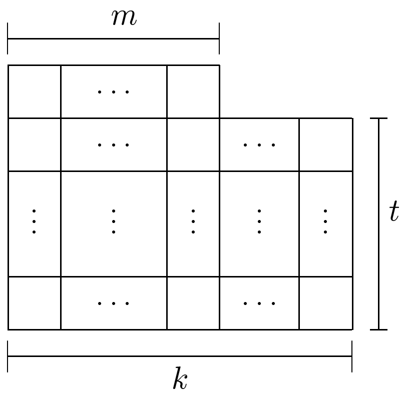
□

We get the following characterization of these special level k fusion coefficients.

Theorem 5.1.8. *Let μ and ν be arbitrary weights on level k, for $0 \leq t \leq N-1$ and $m \leq k$ let λ be the weight associated to the orbit $[m\lambda_1] \times [k\lambda_1]^t$, let μ' be the weight whose corresponding orbit is $[\mu'] = [\mu] \times [k\lambda_1]^t$, and assume that $\hat{\mu} = ((N-1)^{a_{N-1}}, \ldots, 1^{a_1}, 0^{a_0})$. Then we have*

$$M_{[\lambda],[\mu]}^{(k)[\nu]} = \begin{cases} 1, & \text{if } \tilde{\nu} = ((N-1)^{b_{N-1}-m_{N-1}+m_{N-2}}, \ldots, 1^{b_1-m_1+m_0}, 0^{b_0-m_0+m_{N-1}}), \\ & \text{where } m_0, \ldots, m_{N-1} \text{ are integers such that } \sum_{i=0}^{N-1} m_i = m, \\ & 0 \leq m_i \leq b_i \text{ and } b_i = \begin{cases} a_{i-t} & \text{if } i \geq t \\ a_{N-(t-i)} & \text{if } i < t \end{cases} \\ 0, & \text{otherwise} \end{cases}$$

As a summary of the results in this section, more precisely, from Theorem 5.1.7, we get an algorithm to compute fusion coefficients via orbits for Young diagrams of the form



whose corresponding weights are

$$\lambda = \begin{cases} m\lambda_1, & t = 0, \\ (k-m)\lambda_t + m\lambda_{t+1}, & 1 \le t \le N-2, \\ (k-m)\lambda_{N-1}, & t = N-1, \end{cases} \tag{5.6}$$

for $0 \le m \le k$. The corresponding orbits are

$$[\lambda] = \begin{cases} [((t+1)^m, t^{k-m})], & 0 \le t \le N-2, \\ [((N-1)^{k-m}, 0^m)], & t = N-1. \end{cases} \tag{5.7}$$

Example 5.1.9. *For A_3 level 3 ($N = 4$ and $k = 3$), let $\lambda = 2\lambda_3 + \lambda_2$ and let $\mu = 2\lambda_1 + \lambda_2$. The weight λ has the form (5.6) with $t = 2$ and $m = 2$. Theorem 5.1.7 says that we can compute the fusion product $V^\lambda \otimes_3 V^\mu$ via orbits of $\mathbb{Z}_4^3$. The orbits associated to λ and μ are $[\lambda] = [(3,3,2)]$ and $[\mu] = [(2,1,1)]$. To compute their product, we have to remove the redundant equation from the list*

$$\begin{aligned} (3,3,2) &+ (2,1,1) = (1,0,3), \\ (3,3,2) &+ (1,2,1) = (0,1,3), \\ (3,3,2) &+ (1,1,2) = (0,0,0). \end{aligned}$$

The second equation on this list is redundant, therefore we get

$$[\lambda] \times [\mu] = [(3,1,0)] + [(0,0,0)].$$

The weights associated to $[(3,1,0)]$ and $[(0,0,0)]$ are $\nu_1 = \lambda_1 + \lambda_3$ and $\nu_2 = 0$, so we get

$$V^\lambda \otimes_3 V^\mu = V^{\lambda_1+\lambda_3} \oplus V^0.$$

We can verify this result using the characterization (5.1.8). Note that

$$a_3 = 0, \quad a_2 = 1, \quad a_1 = 2, \quad a_0 = 0,$$

so

$$b_3 = a_1 = 2, \quad b_2 = a_0 = 0, \quad b_1 = a_3 = 0, \quad b_0 = a_2 = 1.$$

There are two different sets of integers $\{m_0, m_1, m_2, m_3\}$ satisfying the conditions of Theorem (5.1.8)*, namely:*

$$\{m_0 = 1, m_1 = 0, m_2 = 0, m_3 = 1\} \quad \text{and} \quad \{m'_0 = 0, m'_1 = 0, m'_2 = 0, m'_3 = 2\}.$$

For the set on the left, $\{m_0 = 1, m_1 = 0, m_2 = 0, m_3 = 1\}$, we get

$$\hat{\nu}_1 = (3^{b_3 - m_3 + m_2}, 2^{b_2 - m_2 + m_1}, 1^{b_1 - m_1 + m_0}, 0^{b_0 - m_0 + m_3}) = (3, 1, 0),$$

and for the set $\{m'_0 = 0, m'_1 = 0, m'_2 = 0, m'_3 = 2\}$, we get

$$\hat{\nu}_2 = (3^{b_3 - m'_3 + m'_2}, 2^{b_2 - m'_2 + m'_1}, 1^{b_1 - m'_1 + m'_0}, 0^{b_0 - m'_0 + m'_3}) = (0, 0, 0),$$

which are the representatives in standard form of the orbits we got, and Theorem (5.1.8) *predicts coefficient zero for the rest of the orbits, which matches the result found in this example.*

5.2 What goes wrong with orbits

The orbits product $[a] \times [b] = \sum_{[c] \in \mathcal{O}} M^{(k)[c]}_{[a],[b]}[c]$ does not match Pieri rules for all cases, as we show in this section.

Example 5.2.1. *Consider the weight $\lambda = \lambda_1 + \lambda_2$ of A_2 on level 3. The partition associated to λ is $(\lambda) = (2, 1, 0)$. By applying iterative Pieri rules (Theorem 4.3.5) to compute $S_{(\lambda)} S_{(\lambda)}$ and the fact that $S_{(\lambda)} = h_1 h_2 - h_3$, we get:*

$$\begin{aligned} S_{(\lambda)} S_{(\lambda)} &= S_3 + S_{(3,3)} + 2S_{(\lambda)} + S_{(0)} \\ &= h_3 + S_{(3,3)} + 2S_{(\lambda)} + 1. \end{aligned}$$

From the Dictionary (4.4)*, this translates into the fusion product*

$$V^{\lambda} \otimes_3 V^{\lambda} = V^{3\lambda_1} \oplus V^{3\lambda_2} \oplus 2V^{\lambda_1 + \lambda_2} \oplus V^0. \tag{5.8}$$

This result does not match the result gotten from the orbits point of view. To see this, first note that the orbit corresponding to $\lambda = \lambda_1 + \lambda_2$ in $\mathbb{Z}_3^3$ is the orbit of $[(2, 1, 0)]$, and by a simple calculation of orbits we get that:

$$[(2, 1, 0)] \times [(2, 1, 0)] = [(1, 1, 1)] + [(2, 2, 2)] + 3[(2, 1, 0)] + [(0, 0, 0)].$$

The orbits in this product represent respectively the weights $3\lambda_1$, $3\lambda_2$, $\lambda_1+\lambda_2$ and 0, which are exactly the same weights we got in (5.8), however multiplicities do not all match.

Example 5.2.2. *Consider the weights $\lambda = \lambda_1 + \lambda_2$ and $\mu = 2\lambda_2$ of A_3 on level 3. The partitions associated to λ and μ are $(\lambda) = (2,1,0)$ and $(\mu) = (2,2)$. Using theorem 4.3.5 to compute the fusion product between the Schur polynomials $S_{(\lambda)}$ and $S_{(\mu)}$ and the fact that $S_{(\mu)} = h_2^2 - h_1h_3$, we get:*

$$S_{(\lambda)}S_{(\mu)} = S_{(3,2,2)} + S_{(3,3,1)} + S_{(2,1)} + S_{(1,1,1)}.$$

Using the correspondence between orbits and partitions defined in the last section of Chapter 2 and given more explicitly below in (5.11) we get

$$V^{\lambda} \otimes_3 V^{\mu} = V^{\lambda_1+2\lambda_3} \oplus V^{2\lambda_2+\lambda_3} \oplus V^{\lambda_1+\lambda_2} \oplus V^{\lambda_3}.$$

Now performing the orbit computation $[\lambda] = [(2,1,0)]$ times $[\mu] = [(2,2,0)]$ in $\mathbb{Z}_4^3$, we get:

$$[(2,1,0)] \times [(2,1,0)] = [(3,2,2)] \oplus [(2,1,0)] \oplus [(3,0,0)].$$

The orbits in this product represent, respectively, the weights $2\lambda_2+\lambda_3$, $\lambda_1+\lambda_2$ and λ_3. Observe that the orbit corresponding to the weight $\lambda_1+2\lambda_3$ is missing in the orbit computation.

5.3 How the orbits method can be fixed

Note that the Schur polynomial corresponding to weight $m\lambda_1$ is $S_{(m,0,\ldots,0)} = h_m$, that is, weights that are multiples of the first fundamental weight λ_1 correspond to the homogeneous symmetric polynomials. According to the Jacobi-Trudi determinant (4.2), these polynomials, generate the quotient ring $\Lambda_N/I^{(N,k)} = \mathbb{Q}[x_1, x_2, \ldots, x_N]^{S_N}/I^{(N,k)}$. Therefore for the rest of the weights of level k whose Young diagram is not of the form (5.6), we can compute the fusion products via orbits by alternating iteration using the information from the Jacobi-Trudi determinant. In the following example we show how this process can be done.

Example 5.3.1. *Consider the weight $\lambda = \lambda_1 + \lambda_2$ of A_2 on level 3, note that this weight is not of the form (5.6). If we want to compute the fusion product of V^λ with itself on level 3 ($V^\lambda \otimes_3 V^\lambda$), we may use the fact that $S_\lambda = h_1h_2 - h_3$ and compute:*

$$\begin{aligned} V^\lambda \otimes_3 V^\lambda &= V^\lambda \otimes_3 \left(V^{\lambda_1} \otimes_3 V^{2\lambda_1} - V^{3\lambda_1}\right) \\ &= V^\lambda \otimes_3 \left(V^{\lambda_1} \otimes_3 V^{2\lambda_1}\right) - V^\lambda \otimes_3 V^{3\lambda_1} \\ &= \left(V^\lambda \otimes_3 V^{\lambda_1}\right) \otimes_3 V^{2\lambda_1} - V^\lambda \otimes_3 V^{3\lambda_1} \end{aligned}$$

Now from Theorem 5.1.4 this can be done iteratively via orbits.

To "fix" the orbits method, we construct an algebra $\mathcal{A}(\mathcal{O}(N,k))$ over $\mathbb{Q}$ whose basis is $\mathcal{O}(N,k)$ where the addition is formal addition and we define a new product of orbits $[a]\cdot[b]$ by using the information from the Jacobi-Trudi determinant (4.2). We define this "new product" as follows.

Let $\hat{a}$, $\hat{b}$ be the standard form of two orbits $[a],[b] \in \mathcal{O}$. Let $(\lambda) = \tilde{a}$ and $(\mu) = \tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_{N-1})$ be the conjugates of $\hat{a}$, $\hat{b}$ considered as partitions. The product $S_{(\lambda)}S_{(\mu)}$ can be computed using Pieri rules iteratively with the help of the Jacobi-Trudi determinant, (4.2). Let $H = (h_{\tilde{b}_i - i + j})_{i,j=1}^{N-1} = (h_{i,j})_{i,j=1}^{N-1}$ then $S_{(\mu)} = \det(H) \in \Lambda_N / I^{(N,k)}$, so

$$S_{(\lambda)}S_{(\mu)} = S_{(\lambda)}\det(H). \tag{5.9}$$

By Theorem 5.1.4 we know that each individual product $S_{(\lambda)}h_{\tilde{b}_i - i + j}$ can be computed via orbits by performing the product $[\lambda]\times[(1^{\tilde{b}_i - i + j}, 0^{k - \tilde{b}_i + i - j})]$, provided that $\tilde{b}_i - i + j \leq k$. Therefore, we get that we can multiply any two Schur polynomials via orbits and by the results of this chapter, product (5.9) is equivalent to compute the following orbit product and then translating the results via the Dictionary 5.12

$$[a]\cdot[b] = \begin{cases} [a]\times[b], & \text{if } [b] \text{ is an orbit} \\ & \text{of the form (5.7),} \\ \displaystyle\sum_{\sigma\in S_{N-1}} \operatorname{sign}(\sigma)((([a]\cdot[h_{1,\sigma 1}])\cdot[h_{2,\sigma 2}])\cdots[h_{N-1,\sigma(N-1)}]) & \text{otherwise,} \end{cases} \tag{5.10}$$

where $[h_{i,j}] = [h_{\tilde{b}_i - i + j}]$, $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_{N-1})$ is the conjugate of $\hat{b}$ considered as partition, and

$$[h_{\tilde{b}_i - i + j}] = \begin{cases} [(1^{\tilde{b}_i - i + j}, 0^{k - \tilde{b}_i - i + j})] & \text{if } 0 \leq \tilde{b}_i - i + j \leq k \\ 0 & \text{otherwise,} \end{cases}$$

is of the form (5.7). We extend this product linearly to all $\mathcal{A}(\mathcal{O}(N,k))$.

From Lemma 5.1.1 we know that the the standard form of an S_k-orbit is a partition whose conjugate is a partition contained in the $(N-1) \times k$ rectangle. The explicit correspondence between these two objects is given by

$$[((N-1)^{a_{N-1}}, \dots, 1^{a_1}, 0^{a_0})] \longleftrightarrow \left(\sum_{i=1}^{N-1} a_i, \sum_{i=2}^{N-1} a_i, \dots, a_1 \right). \tag{5.11}$$

This gives a dictionary between the algebras $\mathcal{A}(\mathcal{O}(N,k))$ and $\Lambda_N / I^{(N,k)}$ as follows

$$\begin{aligned} \mathcal{A}(\mathcal{O}(N,k)) &\longleftrightarrow \Lambda_N / I^{(N,k)} \\ [\lambda] &\longleftrightarrow S_{(\lambda)}, \\ [h_m] = [(1^m, 0^{k-m})] &\longleftrightarrow h_m, \\ [\lambda] \cdot [\mu] &\longleftrightarrow S_{(\lambda)} S_{(\mu)}, \end{aligned} \tag{5.12}$$

where the left hand side is indexed by orbits and the right hand side is indexed by partitions and the correspondence (5.11) allows us to move from left to right.

According to the results in this chapter, we get the product (5.10) after translating to Schur functions using Dictionary 5.12, matching the associative product $S_{(\lambda)} S_{(\mu)}$. Therefore (5.10) is an associative product. Thus (5.10) gives a structure of associative algebra to $\mathcal{A}(\mathcal{O}(N,k))$ and this algebra is isomorphic to $\Lambda_N / I^{(N,k)} \cong \mathcal{F}(A_{N-1}, k)$.

Example 5.3.2. *Continuing with Example 5.3.1 we want to compute the fusion product $V^\lambda \otimes_3 V^\lambda$ where $\lambda = \lambda_1 + \lambda_2$ of A_2 on level 3.*

The orbit corresponding to λ is $[\lambda] = [(2,1,0)]$ and the conjugate of $\hat{\lambda}$ is $\tilde{b} = (2,1)$. Hence Equation (5.10) gives

$$[(2,1,0)] \cdot [(2,1,0)] = [(2,1,0)] \cdot \begin{vmatrix} [h_{\tilde{b}_1}] & [h_{\tilde{b}_1+1}] \\ [h_{\tilde{b}_2 - 1}] & [h_{\tilde{b}_2}] \end{vmatrix}.$$

Since $\tilde{b}_1 = 2$ *and* $\tilde{b}_2 = 1$, *we get*

$$\begin{aligned}[h_{\tilde{b}_1}] &= [(1,1,0)],\\ [h_{\tilde{b}_1+1}] &= [(1,1,1)],\\ [h_{\tilde{b}_2}] &= [(1,0,0)],\\ [h_{\tilde{b}_{-1}}] &= [(0,0,0)].\end{aligned}$$

Therefore we get

$$\begin{aligned}[(2,1,0)]\cdot[(2,1,0)] = [(2,1,0)]\cdot &\begin{vmatrix}[(1,1,0)] & [(1,1,1)]\\ [(0,0,0)] & [(1,0,0)]\end{vmatrix}\\ &= [(2,1,0)]\cdot([(1,1,0)]\cdot[(1,0,0)]) - [(2,1,0)]\cdot[(1,1,1)].\end{aligned}$$

From Example 3.0.5 we know that

$$[(2,1,0)]\cdot[(1,1,0)] = [(2,1,0)]\times[(1,1,0)] = [(2,2,1)] + [(1,1,0)] + [(2,0,0)]$$

and it is readily checked that

$$\begin{aligned}[(2,2,1)]\cdot[(1,0,0)] &= [(2,2,1)]\times[(1,0,0)] = [(2,2,2)] + [(2,1,0)],\\ [(1,1,0)]\cdot[(1,0,0)] &= [(1,1,0)]\times[(1,0,0)] = [(1,1,1)] + [(2,1,0)],\\ [(2,0,0)]\cdot[(1,0,0)] &= [(2,0,0)]\times[(1,0,0)] = [(2,1,0)] + [(0,0,0)]\end{aligned}$$

and

$$[(2,1,0)]\cdot[(1,1,1)] = [(2,1,0)]\times[(1,1,1)] = [(2,1,0)].$$

Therefore we get

$$[(2,1,0)]\cdot[(2,1,0)] = [(2,2,2)] + [(1,1,1)] + 2[(2,1,0)] + [(0,0,0)],$$

which translates into the fusion product

$$V^{\lambda_1+\lambda_2}\otimes_3 V^{\lambda_1+\lambda_2} = V^{3\lambda_2}\oplus V^{3\lambda_1}\oplus 2V^{\lambda_1+\lambda_2}\oplus V^0.$$

5.4 Rank-level duality from the orbits point of view

The sub-algebra of simple currents of $\mathcal{F}(A_{N-1}, k)$ is the sub-algebra with basis $\{h_k^t \mid 0 \leq t \leq N-1\}$. The orbit corresponding to the partition $(k, 0, ..., 0)$ is the orbit $[(1^k)]$ and by multiplying this orbit by itself iteratively we get that $[(1^k)]^t = [(t^k)]$. Therefore the orbits corresponding to the simple currents of $\mathcal{F}(A_{N-1}, k)$ have the form $[(t^k)]$. Now, let $[b] = [(t^k)]$ be the orbit of a simple current and $[a] = [(m_1, ..., m_k)] \in \mathcal{O}(N, k)$. Then the product $[a] \cdot [b]$ can be easily described as

$$[a] \cdot [b] = [(m_1 + t, ..., m_k + t)],$$

where every entry is considered modulo N. By abuse of notation we are going to denote the orbit on the right hand side by $[a + t]$.

This product gives an action of $\mathbb{Z}_N$ on $\mathcal{O}(N, k)$. This action breaks the set $\mathcal{O}(N, k)$ into "orbits", that we call SC-orbits, to avoid confusion with S_k-orbits of $\mathbb{Z}_N^k$ and as an abbreviation for simple current orbits.

Note that for $0 \leq m \leq k$ the product of any element in the SC-orbit of $[(1^m, 0^{k-m})]$ by any SC-orbit is independent of the choice of representative. In fact, let $0 \leq t, t_1 \leq N-1$ and $[a] \in \mathcal{O}(N, k)$, then $[a + t_1]$ is in the SC-orbit of $[a]$ and $[((t+1)^m, t^{k-m})]$ is in the SC-orbit of $[(1^m, 0^{k-m})]$ and is one of the orbits of the form (5.7), therefore

$$\begin{aligned} [a + t_1] \cdot [((t+1)^m, t^{k-m})] =& [a] \cdot [(t_1^k)] \cdot [(1^m, 0^{k-m})] \cdot [(t^k)] \\ =& [a] \cdot [(1^m, 0^{k-m})] \cdot [((t + t_1)^k)]. \end{aligned} \tag{5.13}$$

This defines a product on the set of SC-orbits of $\mathcal{O}(N, k)$, and it is equivalent to the product in the algebra $\mathcal{A}'(\mathcal{O}(N, k)) = \mathcal{A}(\mathcal{O}(N, k))/SC$, where SC is the ideal of $\mathcal{A}(\mathcal{O}(N, k))$ generated by $[(t^k)] - [(0^k)]$. This algebra is isomorphic to $\mathcal{F}'(A_{N-1}, k)$.

To see the rank-level duality from the orbits point of view, first observe that any SC-orbit of $\mathcal{O}(N, k)$ has a representative of the form $[((N-1)^{a_{N-1}}, ..., 0^{a_0})]$ where $a_0 > 0$, since for any orbit $[(m_1, ..., m_k)] \in \mathcal{O}(N, k)$ we can add (t^k) enough times to make one of the entries zero.

Example 5.4.1. *Consider the set* $\mathcal{O}(4,3)$. *The SC-orbit of the element* $[(2,2,1)]$ *is the set* $\{[(2,2,1)],[(3,3,2)],[(3,0,0)],[(1,1,0)]\}$. *Note that in fact this SC-orbit contains two elements with at least one zero entry.*

Let $\mathcal{O}'(N,k) = \{[a] = [(a_1,\ldots,a_k)] \in \mathcal{O}(N,k) \mid a_i = 0 \text{ for some } i, 1 \leq i \leq k\}$, the set of S_k-orbits of $\mathbb{Z}_k^N$ with at least one zero entry. We have a one-to-one correspondence given by

$$\begin{aligned} \mathcal{O}'(N,k) &\longrightarrow \mathcal{O}'(k,N) \\ [((N-1)^{a_{(N-1)}},...,1^{a_1},0^{a_0})] &\longmapsto \left[\left(\sum_{i=1}^{N-1} a_i, \sum_{i=2}^{N-1} a_i, ..., a_{(N-1)}, 0\right)\right], \end{aligned}$$

where $a_0 > 0$ and $\sum_{i=0}^{N-1} a_i = k$.

This map sends SC-orbits of $\mathcal{O}(N,k)$ to SC-orbits of $\mathcal{O}(k,N)$, and since the map is defined by conjugation of the orbits considered as partitions, it follows that the map is just a different way of defining the rank-level duality map defined in Section 4.4 and Equation (4.13).

Chapter 6

Application to tensor product

6.1 Classical Pieri rules

Irreducible modules V^λ for the finite dimensional Lie algebra $\mathfrak{g} = sl_N$, are indexed by weights $\lambda \in P^+$ and form a basis of an associative algebra over $\mathbb{Q}$, called the tensor algebra and denoted by $T(A_{N-1})$, with associative product of V^λ and V^μ defined by the tensor product of the two modules $V^\lambda \otimes V^\mu$. This product is a well defined product in $T(A_{N-1})$ since given modules V^μ and V^λ for sl_N, with μ and $\lambda \in P^+$, the tensor product $V^\mu \otimes V^\lambda$ is completely reducible and it is given by

$$V^\lambda \otimes V^\mu = \sum_{\nu \in P^+} Mult_{\lambda,\mu}(\nu) V^\nu \tag{6.1}$$

where $Mult_{\lambda,\mu}(\nu)$ is given by the Racah-Speiser formula (2.6).

The Littlewood-Richardson coefficients are the structure constants for the algebra of symmetric polynomials Λ_N with respect to the basis given by Schur polynomials indexed by partitions of length at most N. If (μ) and (λ) are partitions of length at most N and $S_{(\mu)}$ and $S_{(\lambda)}$ are the Schur polynomials as defined in (4.2) (or in (4.3)), then we have

$$S_{(\lambda)} S_{(\mu)} = \sum_{(\nu) \text{ is a partition of length at most } N} c^{(\nu)}_{(\lambda),(\mu)} S_{(\nu)}. \tag{6.2}$$

There is a relation between these two algebras. In fact, it is well known that there is an isomorphism between the tensor algebra $T(A_{N-1})$ and the quotient algebra $\Lambda_N/\langle S_{(1^N)}-1\rangle$, given by

$$\begin{aligned} \Lambda_N/\langle S_{(1^N)}-1\rangle &\longleftrightarrow T(A_{N-1}) \\ S_{(\mu)} &\longleftrightarrow V^{\mu} \\ S_{(\lambda)}S_{(\mu)} = \sum_{(\nu)} c^{(\nu)}_{(\lambda),(\mu)} S_{(\nu)} &\longleftrightarrow V^{\mu}\otimes V^{\lambda} = \bigoplus_{\nu} Mult_{\lambda,\mu}(\nu) V^{\nu}, \end{aligned} \tag{6.3}$$

where the left hand side is indexed by partitions $(\mu)=(\mu_1,...,\mu_N)$ of length at most N, and the right hand side is indexed by weights $\mu=\sum_{j=1}^{N-1}(\mu_j-\mu_{j+1})\lambda_j \in P^+$ where P^+ is the set of dominant integral weights defined in (2.7). The correspondence (2.10) allows one to move from one side to the other and $c^{(\nu)}_{(\lambda),(\mu)} = Mult_{\lambda,\mu}(\nu)$.

Particular cases of the product (6.2) can be computed using Pieri rules. The following results are known as classical Pieri rules.

Theorem 6.1.1. *(Classical Pieri rules for multiplication by h_m)*

Let (μ) be a partition of length at most N and m a positive integer, then in $\Lambda_N/\langle S_{(1^N)}-1\rangle$ we have

$$S_{(\mu)}h_m = \sum_{(\nu)/(\mu) \textit{ is an m-row strip}} S_{(\nu)}.$$

Theorem 6.1.2. *(Classical Pieri rules for multiplication by e_m)*

Let (μ) be a partition of length at most N and $0\le m\le N$ an integer, then in $\Lambda_N/\langle S_{(1^N)}-1\rangle$ we have

$$S_{(\mu)}e_m = \sum_{(\nu)/(\mu) \textit{ is an m-column strip}} S_{(\nu)}.$$

It is also known that the Littlewood-Richardson coefficients $c^{(\nu)}_{(\mu),(\lambda)}$ are at most one when $(\lambda)=(m)$ is a partition with a single part and when $(\lambda)=(1^m)$ is a partition with only ones. The characterization is given by

$$c^{(\nu)}_{(\mu),(m)} = \begin{cases} 1, & \text{if there exists a partition } (\bar{\nu}) \text{ of length at most N,} \\ & \text{such that } (\bar{\nu})/(\mu) \text{ is an m-row strip and } (\bar{\nu})\sim(\nu), \\ 0, & \text{otherwise,} \end{cases} \tag{6.4}$$

where $\sim$ is the equivalence relation defined in Equation (2.12), and

$$c^{(\nu)}_{(\mu),(1^m)} = \begin{cases} 1, & \text{if there exists a partition } (\bar{\nu}) \text{ of length at most N,} \\ & \text{such that } (\bar{\nu})/(\mu) \text{ is an m-column strip and } (\bar{\nu}) \sim (\nu), \\ 0, & \text{otherwise.} \end{cases} \tag{6.5}$$

We are going to use Equation (6.4) to prove that we can compute the weight decomposition of $V^{\mu} \otimes V^{m\lambda_1}$ via orbits, but since there is no level k condition for tensor products, the sum $\sum_{j=1}^{N-1} a_j$ and m can be arbitrarily large. Therefore, our 1-1 correspondence between weights and orbits, given by (2.9), has to be modified, and we also have to modify our group G.

6.2 S_∞-orbits of $\bigoplus_{i=1}^{\infty} \mathbb{Z}_N$

Let $\bar{G} = \bar{G}_N = \bigoplus_{i=1}^{\infty} \mathbb{Z}_N = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z}_N, x_j = 0 \text{ for } j >> 0\}$ and let S_∞, the group of permutations of $\{1, 2, \dots\}$ which fix all but finitely many numbers, act on $\bar{G}_N$ by permuting the coordinates. For $x \in \bar{G}_N$, define the element in standard form S_∞-equivalent to x to be

$$\hat{x} = ((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0, ...)$$

where i occurs a_i times in x.

There is a 1-1 correspondence between dominant weights of A_{N-1} and S_∞-orbits of $\bar{G}$ as follows:

$$a_1\lambda_1 + ... + a_{N-1}\lambda_{N-1} \in P^+ \leftrightarrow [((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0, ...)], \tag{6.6}$$

where the bracket on the right means the S_∞-orbit of the element in standard form.

The correspondence between partitions of length at most N and weights of A_{N-1} given by (2.10) and the correspondence (6.6), gives us a correspondence between partitions of length at most N and orbits of $\bar{G}$. In this chapter will use the same notation as before for weights and partitions, that is, if μ is a weight, then (μ)

denotes its corresponding partition, but for this chapter, $[\mu]$ will denote the S_∞-orbit in $\bar{G}$ corresponding to the weight μ and the representative in standard form of $[\mu]$ in $\bar{G}$ will be denoted by $\hat{\mu}$. The set of S_∞-orbits of $\bar{G}$ will be denoted $\mathcal{O}_\infty = \mathcal{O}_\infty(N)$.

Now we extend the definition of the set $T([a],[b],[c])$ to S_∞-orbits $[a]$, $[b]$ and $[c]$ of $\bar{G}$:

$$T([a],[b],[c]) = \{(x,y,z) \in [a] \times [b] \times [c] \mid x + y = z\},$$

and define $M_{[a],[b]}^{[c]}$ as the number of S_∞-orbits of $T([a],[b],[c])$.

The number $M_{[a],[b]}^{[c]}$ can be also used to define a product of S_∞-orbits of $\bar{G}_N$ as follows

$$[a] \times [b] = \sum_{[c] \in \mathcal{O}_\infty} M_{[a],[b]}^{[c]} [c].$$

The procedure to compute the number $M_{[a],[b]}^{[c]}$ is similar to the one we used to compute the number $M_{[a],[b]}^{(k)[c]}$. In fact, since elements in $\bar{G}$ have a finite number of non-zero entries, we can follow exactly the same procedure to compute $M_{[a],[b]}^{(k)[c]}$, with $k >> 0$, as follows.

Definition 6.2.1. *Let $[a]$, $[b] \in \mathcal{O}_\infty$, and assume that $\hat{a} = ((N-1)^{a_{N-1}}, ..., 1^{a_1}, 0, ...)$ $\hat{b} = ((N-1)^{b_{N-1}}, ..., 1^{b_1}, 0, ...)$, and set k to be any integer such that $k \geq \sum_{i=1}^{N-1}(a_i + b_i)$. Let $\{y_1, \dots, y_t\} \subset [b]$ be the set of orbits of $\hat{b}$ obtained by letting S_k act on the first k entries of $\hat{b}$, and for $1 \leq i \leq t$ set:*

$$z_i = \hat{a} + y_i. \tag{6.7}$$

*We say that the equation $z_j = \hat{a} + y_j$ in the list (3.5) is **redundant**, if for some $i < j$ and $\sigma \in S_k$ we have*

$$\sigma\hat{a} = \hat{a}, \qquad \sigma y_j = y_i \quad \textit{and} \quad \sigma z_j = z_i,$$

that is, if the triples $(\hat{a}, y_i, z_i)$ and $(\hat{a}, y_j, z_j)$ are in the same S_k-orbit of $T([a],[b],[z_i])$.

Then it is easily seen that

$$M_{[a],[b]}^{[c]} \text{ equals the number of non-redundant equations of the form (6.7) .} \tag{6.8}$$

From this, it follows that $M_{[a],[b]}^{[c]} = M_{[a],[b]}^{(k)[c]}$, for $k >> 0$.

Example 6.2.2. *Let $a = (2, 1, 0, \dots)$ and $b = (1, 1, 0, \dots)$ be elements in $\bar{G}_3$. Then we have that $\hat{a} = (2, 1, 0, \dots)$ and $\hat{b} = (1, 1, 0, \dots)$. So the least k we can choose to compute the orbit product $[a] \times [b]$ is $k = 4$. By letting S_4 act on the first four entries of $\hat{b}$ we get the subset of $[b]$*

$$\begin{aligned}\{\hat{b} =& (1, 1, 0, \dots), (1, 0, 1, 0, \dots), (1, 0, 0, 1, 0, \dots), \\ &(0, 1, 1, 0, \dots), (0, 1, 0, 1, 0, \dots), (0, 0, 1, 1, 0, \dots)\}.\end{aligned}$$

Now we have to remove all redundancies from the list:

$$\begin{aligned}&(2, 1, 0, \dots) + (1, 1, 0, \dots) = (0, 2, 0, \dots), \\ &(2, 1, 0, \dots) + (1, 0, 1, 0, \dots) = (0, 1, 1, 0, \dots), \\ &(2, 1, 0, \dots) + (1, 0, 0, 1, 0, \dots) = (0, 1, 0, 1, 0, \dots), \\ &(2, 1, 0, \dots) + (0, 1, 1, 0, \dots) = (2, 2, 1, 0, \dots), \\ &(2, 1, 0, \dots) + (0, 1, 0, 1, 0, \dots) = (2, 2, 0, 1, 0, \dots), \\ &(2, 1, 0, \dots) + (0, 0, 1, 1, 0, \dots) = (2, 1, 1, 1, 0, \dots).\end{aligned}$$

The third and the fifth equations are redundant in that list. So we get:

$$\begin{aligned}[(2, 1, 0, \dots)] \times [(1, 1, 0, \dots)] =& [(2, 2, 1, 0, \dots)] + [(1, 1, 0, \dots)] \\ &+ [(2, 0, 0, \dots)] + [(2, 1, 1, 1, 0, \dots)].\end{aligned} \tag{6.9}$$

Later we will see how this corresponds to a tensor product decomposition for A_2 of $V^{\lambda_1+\lambda_2} \otimes V^{2\lambda_1}$.

Example 6.2.3. *Now let $a = (2, 2, 1, 0, \dots)$ and $b = (1, 0, 0, \dots)$ be elements in $\bar{G}_3$. Then we have that $\hat{a} = (2, 2, 1)$ and $\hat{b} = (1, 0, 0, \dots)$. We can take $k = 4$ to compute $[a] \times [b]$. By letting S_4 act on the first four entries of $\hat{b}$ we get the subset of $[b]$*

$$\{(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, 0, \dots), (0, 0, 0, 1, 0, \dots)\}.$$

Then we remove all redundancies from the list:

$$(2,2,1,0,\dots)+(1,0,0,\dots)=(0,2,1,0,\dots),$$
$$(2,2,1,0,\dots)+(0,1,0,\dots)=(2,0,1,0,\dots),$$
$$(2,2,1,0,\dots)+(0,0,1,0,\dots)=(2,2,2,0,\dots),$$
$$(2,2,1,0,\dots)+(0,0,0,1,0,\dots)=(2,2,1,1,0,\dots).$$

We can see that the second equation is redundant, so we can remove it from the list and we get

$$[(2,2,1,0,\dots)]\times[(1,0,\dots)]=[(2,2,2,0,\dots)]+[(2,1,0,\dots)]+[(2,2,1,1,0,\dots)].$$

Theorem 6.2.4. *Let $[a],[c]\in\mathcal{O}$ and $[b]=[(1^m,0,\dots)]$. Suppose that $M^{[c]}_{[a],[b]}\neq 0$ then $M^{[c]}_{[a],[b]}=1$.*

Proof. Since for $k>>0$ we have $M^{[c]}_{[a],[b]}=M^{(k)[c]}_{[a],[b]}$, the proof follows from Theorem 3.0.9. $\square$

We also get the following characterization of orbits products similar to Corollary 3.0.10 in Chapter 3.

Corollary 6.2.5. *Let $[a]\in\mathcal{O}$, assume that $\hat{a}=((N-1)^{a_{N-1}},...,1^{a_1},0,\dots)$ and let $[b]=[(1^m,0,\dots)]$. Then for $[c]\in\mathcal{O}$ we have*

$$M^{[c]}_{[a],[b]}=\begin{cases}1, & \text{if } \hat{c}=((N-1)^{a_{N-1}-m_{N-1}+m_{N-2}},...,1^{a_1-m_1+m_0},0,\dots),\\ & \text{for some integers } m_0,\ m_1,...,m_{N-1} \text{ such that } \sum_{i=0}^{N-1}m_i=m\\ & \text{and}\quad 0\leq m_i\leq a_i\quad \text{, for}\quad 1\leq i\leq N-1,\\ 0, & \text{otherwise.}\end{cases}$$

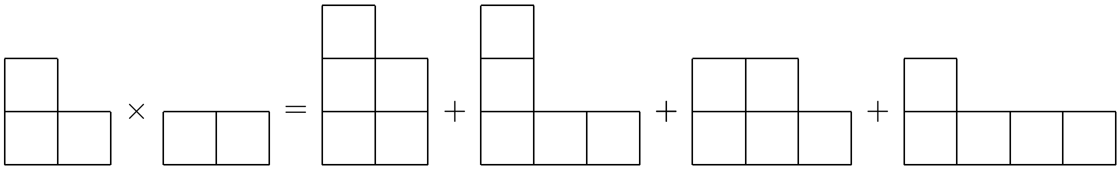
It is well known that for $k>>0$ the coefficients $c^{(\nu)}_{(\mu),(\lambda)}=N^{(k)(\nu)}_{(\mu),(\lambda)}$, see [Fe], Theorem 6.1. Then we have the following result.

Theorem 6.2.6. *For $1\leq m\in\mathbb{Z}$ let $\lambda=m\lambda_1$ be a multiple of the first fundamental weight for A_{N-1}, $\mu=a_1\lambda_1+...+a_{N-1}\lambda_{N-1}$ any other dominant weight and $[\lambda]$ and*

$[\mu]$ *their corresponding orbits in* $\bar{G}$. *Then* $c^{(\nu)}_{(\mu),(\lambda)} = M^{[\nu]}_{[\mu],[\lambda]}$ *for any dominant weight* ν. *In other words, the product of the* S_∞*-orbits* $[\lambda] \times [\mu]$ *matches classical Pieri rules for the multiplication* $S_{(\mu)} h_m$, *Theorem 6.1.1.*

Proof. For $k >> 0$ we have $M^{[\nu]}_{[\mu],[\lambda]} = M^{(k)[\nu]}_{[\mu],[\lambda]}$ and $c^{(\nu)}_{(\mu),(\lambda)} = N^{(k)(\nu)}_{(\mu),(\lambda)}$. Now from Theorem 5.1.4 we have that $M^{(k)[\nu]}_{[\mu],[\lambda]} = N^{(k)(\nu)}_{(\mu),(\lambda)}$ from where the theorem follows. □

Example 6.2.7. *Let* $(\mu) = (2,1)$ *and* $N = 3$. *Using Theorem 6.1.1 to compute* $S_{(\mu)} h_2$ *we get*



that is, in the algebra $\Lambda_3/\langle S_{(1^3)} - 1\rangle$, *we have that:*

$$S_{(2,1)} h_2 = S_{(2,2,1)} + S_{(3,1,1)} + S_{(3,2)} + S_{(4,1)}.$$

Now, the weight associated to the partition $(\mu) = (2,1)$ *is* $\mu = \lambda_1 + \lambda_2$ *and the weight associated to* $h_2 = S_{(2,0,0)}$ *is* $2\lambda_1$. *Then the above equation translates into the tensor product in* $T(A_2)$

$$V^{\lambda_1+\lambda_2} \otimes_3 V^{2\lambda_1} = V^{\lambda_2} \oplus V^{2\lambda_1} \oplus V^{\lambda_1+2\lambda_2} \oplus V^{3\lambda_1+\lambda_2}.$$

Under the correspondence (2.9) *we have that the orbits corresponding to the weights* $\lambda_1 + \lambda_2$, $2\lambda_1$, λ_2, $\lambda_1 + 2\lambda_2$ *and* $3\lambda_1 + \lambda_2$ *are respectively [(2,1,0,...)], [(1,1,0,...)], [(2,0,...)], [(2,2,1,0...)] and [(2,1,1,1,0...)]. Looking at Example 6.2.2 we see that this tensor product agrees with the orbit calculation from that example.*

6.3 Weight space decomposition of modules

In this section we are going to describe how we can use Young tableaux to find the weight decomposition of an irreducible module for sl_N.

We start by noticing that the map (2.11) between integral dominant weights of A_{N-1} and partitions of length at most N can be extended to a map from the set of sequences of non-negative integers of length N to the weight lattice P of A_{N-1}

$$(\mu) = (\mu_1, \ldots, \mu_N) \longmapsto \mu = \sum_{i=1}^{N-1} (\mu_i - \mu_{i+1})\lambda_i. \tag{6.10}$$

The map is not one-to-one but it has a right inverse given by

$$\mu = \sum_{i=1}^{N-1} a_i\lambda_i \longmapsto (\mu) = \left(c + \sum_{i=1}^{N-1} a_i, c + \sum_{i=2}^{N-1} a_i \ldots, c + a_{N-1}, c \right),$$

where $c = \min \left\{ \sum_{i=1}^{N-1} a_i, \sum_{i=2}^{N-1} a_i, \ldots, a_{N-1}, 0 \right\}$.

Let $V = V^\lambda$ be an irreducible finite dimensional module for sl_N where $\lambda = \sum_{i=1}^{N-1} a_i\lambda_i \in P^+$. Let

$$V^\lambda = \bigoplus_{\beta \in P} \mathrm{Mult}_\lambda(\beta) V_\beta^\lambda$$

be the weight space decomposition of V^λ where

$$V_\beta^\lambda = \{x \mid h \cdot x = \beta(h)x, \text{ for all } h \in \mathfrak{h}\}.$$

Let $(\lambda) = \left(\sum_{i=1}^{N-1} a_i, \ldots, a_{N-1}, 0 \right)$ be a partition associated to λ. The set of Young tableaux of shape (λ) filled with numbers from 1 to N is in 1-1 correspondence with the set of weight spaces of V^λ including multiplicities. The correspondence is as follows:

The number of Young tableaux of shape (λ) and content $(\mu) = (\mu_1, \ldots, \mu_N)$ is equal to $\mathrm{Mult}_\lambda(\beta)$ where $\beta = \sum_{i=1}^{N-1} (\mu_i - \mu_{i+1})\lambda_i$.

Example 6.3.1. *Consider the module V^λ for A_2, where $\lambda = \lambda_1 + \lambda_2$, the well known adjoint representation of sl_3. The partition associated to λ is $(\lambda) = (2, 1, 0)$. Below we show the set of Young tableaux of shape (λ) filled with numbers from 1 to 3, and underneath the each tableau we have the content and the weight associated to the content.*

$$\begin{array}{cccc}
\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} &
\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{1} \\ \hline\end{array} &
\begin{array}{|c|c}\hline 2 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} &
\begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array} \\
(2,1,0) & (2,0,1) & (1,2,0) & (1,1,1) \\
\lambda_1 + \lambda_2 & 2\lambda_1 - \lambda_2 & -\lambda_1 + 2\lambda_2 & 0
\end{array}$$

$$
\begin{array}{cccc}
\begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} &
\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array} &
\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline\end{array} &
\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \\
(1,1,1) & (1,0,2) & (0,2,1) & (0,1,2) \\
0 & \lambda_1 - 2\lambda_2 & -2\lambda_1 + \lambda_2 & -\lambda_1 - \lambda_2
\end{array}
$$

Therefore we have that (leaving off the superscript $\lambda_1 + \lambda_2$ from the weight spaces on the right hand side)

$$V^{\lambda_1+\lambda_2} = V_{\lambda_1+\lambda_2} \oplus V_{2\lambda_1-\lambda_2} \oplus V_{-\lambda_1+2\lambda_2} \oplus 2V_0 \oplus V_{\lambda_1-2\lambda_2} \oplus V_{-2\lambda_1+\lambda_2} \oplus V_{-\lambda_1-\lambda_2}.$$

Now, we discuss how we can use this to implement the Racah-Speiser algorithm.

The Weyl group $W \cong S_N$ of sl_N acts on sequences of length N as permutations, with the simple reflections acting as transpositions

$$r_i(\mu_1, \ldots, \mu_N) = (\mu_1, \ldots, \mu_{i+1}, \mu_i, \ldots, \mu_N), \qquad i = 1, ..., N-1. \tag{6.11}$$

This action allows one to get a version of the Racah-Speiser algorithm from the Young tableau point of view as follows.

Let V^λ and V^μ be irreducible finite dimensional modules of type A_{N-1}, (λ) and (μ) the partitions associated to λ and μ respectively. The decomposition of the tensor product $V^\lambda \otimes V^\mu$ into irreducible modules can be computed by doing the following.

Step 1. List all contents of the Young tableaux of shape (λ) whose fillings are with numbers from 1 to N.

Step 2. Add $(\mu) + (\rho)$ to all contents from step 1, where $(\rho) = (N-1, N-2, \ldots, 1, 0)$.

Step 3. Apply the action of the Weyl group (6.11) to all sequences that are not in standard form from step 2 to write them in standard form, with positive multiplicity if the number of transpositions is even and negative multiplicity if the number of transpositions is odd and we drop the sequences that are fixed by any transposition.

Step 4. Subtract (ρ) from all partitions left in step 3 and use the correspondence (6.10) to get the weights λ associated to the partitions from step 3. The direct sum of the irreducible modules indexed by these weights equals the tensor product decomposition of $V^\lambda \otimes V^\mu$.

Example 6.3.2. *Let* $V^{\lambda_1+\lambda_2}$ *and* $V^{2\lambda_1}$ *be irreducible modules for* A_2*. The weights of* $V^{\lambda_1+\lambda_2}$ *were given in Example 6.3.1 as well as all contents of tableaux of shape* $(\lambda_1+\lambda_2) = (2,1)$*. If we add* $(2\lambda_1) + (\rho) = (2,0,0) + (2,1,0) = (4,1,0)$ *to all these weights we get the sequences*

$$(6,2,0), \quad (6,1,1), \quad (5,3,0), \quad 2(5,2,1), \quad (5,1,2), \quad (4,3,1), \quad (4,2,2),$$

where the coefficient 2 in front of $(5,2,1)$ *is the number of tableaux of shape* $(\lambda_1+\lambda_2) = (2,1)$ *and that content.*

The sequences $(6,1,1)$ *and* $(4,2,2)$ *are fixed by the action of* r_2*, therefore they do not count for the tensor product. The weight* $(5,1,2)$ *is not in standard form, but if we apply* r_2 *to it, we get* $r_2 \cdot (5,1,2) = (5,2,1)$*, so it reduces the multiplicity of* $(5,2,1)$ *to 1. So we are left with the sequences*

$$(6,2,0), \qquad (5,3,0), \qquad (5,2,1), \qquad (4,3,1).$$

By subtracting $(\rho) = (2,1,0)$ *from these weights we get*

$$(4,1,0), \quad (3,2,0), \quad (3,1,1), \quad (2,2,1)$$

and using the map (6.10) *we get that the weights associated to these sequences are*

$$3\lambda_1 + \lambda_2, \qquad \lambda_1 + 2\lambda_2, \qquad 2\lambda_1, \qquad \lambda_2.$$

Therefore the tensor product decomposition of $V^{\lambda_1+\lambda_2} \otimes V^{2\lambda_1}$ *is given by*

$$V^{\lambda_1+\lambda_2} \otimes V^{2\lambda_1} = V^{3\lambda_1+\lambda_2} \oplus V^{\lambda_1+2\lambda_2} \oplus V^{2\lambda_1} \oplus V^{\lambda_2}.$$

The advantage of this method over the geometrical one, presented in section 2.4, is that we can compute tensor product decompositions by hand for ranks higher than 2 on a piece of paper.

Now we present a version of the Kac-Walton algorithm from the tableau point of view. For a given $k > 0$, the affine reflection r_0 acts on the finite dimensional weight lattice P, and therefore on sequences of non-negative numbers of length at most N. This action is defined by

$$r_0 \cdot (a_0, \ldots, a_{N-1}) = \begin{cases} (k+N+a_{N-1}, a_1, \ldots, a_{N-2}, a_0 - k - N), & \\ \qquad\qquad\qquad\qquad \text{if} \quad a_0 - a_{N-1} \geq k+N, & \\ (k+N+a_{N-1}+c, a_1+c, \ldots, a_{N-2}+c, a_0-k-N+c), & \\ \qquad\qquad\qquad\qquad \text{if} \quad a_0 - a_{N-1} < k+N, & \end{cases} \tag{6.12}$$

where $c = k + N - a_0$ if $a_0 - k - N < 0$ and $c = 0$ otherwise.

With the level k action of the affine Weyl group $\widehat{W}$ defined this way, we get the following algorithm for fusion products. Note that the fundamental region for the action of $\widehat{W}$ defined by (6.11) and (6.12) on length N sequences of non-negative integers is the set of partitions $(a_0, \ldots, a_{N-1})$ of length at most N such that $a_0 - a_{N-1} \leq N + k$. We denote this set by H_{N+k}. That is

$$H_{N+k} = \{(a_0, \ldots, a_{N-1}) \in \mathbb{Z}^N_{\geq 0} \mid a_0 \geq \cdots \geq a_{N-1} \text{ and } a_0 - a_{N-1} \leq N + k\}.$$

Let V^λ and V^μ be irreducible highest weight modules of type $A^{(1)}_{N-1}$ on level k, (λ) and (μ) the partitions associated to λ and μ respectively. The level k fusion product $V^\lambda \otimes_k V^\mu$ into irreducible modules can be computed by doing the following.

Step 1. List all contents of the Young tableaux of shape (λ) whose fillings are with numbers from 1 to N.

Step 2. Add $(\mu) + (\rho)$ to all contents from step 1, where $(\rho) = (N-1, N-2, \ldots, 1, 0)$.

Step 3. Apply the action of the affine Weyl group defined by (6.11) and (6.12) to all sequences from step 2 to get them into the fundamental region H_{N+k}, with positive multiplicity if the number of reflections is even and negative multiplicity if the number of reflections is odd and drop the sequences that are fixed by any reflection.

Step 4. Subtract (ρ) from all partitions left in step 3 and use the correspondence (6.10) to get the weights λ associated to the partitions from step 3. The direct sum of the irreducible modules indexed by these weights equals the level k fusion product $V^\lambda \otimes_k V^\mu$.

Example 6.3.3. *Let $V^{\lambda_1+\lambda_2}$ and $V^{2\lambda_1}$ be irreducible modules for $A_2^{(1)}$ and $k = 2$. The weights of $V^{\lambda_1+\lambda_2}$ were given in Example 6.3.1 as well as all contents of tableaux of shape $(\lambda_1 + \lambda_2) = (2, 1)$. If we add $(2\lambda_1) + (\rho) = (2, 0, 0) + (2, 1, 0) = (4, 1, 0)$ to all these weights we get the sequences*

$$(6, 2, 0), \quad (6, 1, 1), \quad (5, 3, 0), \quad 2(5, 2, 1), \quad (5, 1, 2), \quad (4, 3, 1), \quad (4, 2, 2),$$

where the coefficient 2 in front of $(5, 2, 1)$ is the number of tableaux of shape $(\lambda_1+\lambda_2) = (2, 1)$ and that content.

The sequences $(6, 1, 1)$ and $(4, 2, 2)$ are fixed by the action of r_2, and $(5, 3, 0)$ is fixed by r_0, therefore they do not count for the fusion product. The weights $(5, 1, 2)$ and $(6, 2, 0)$ are outside the fundamental region H_5. Applying r_2 to $(5, 1, 2)$, we get $r_2 \cdot (5, 1, 2) = (5, 2, 1)$, and applying r_0 to $(6, 2, 0)$ we get $r_0 \cdot (6, 2, 0) = (5, 2, 1)$. Therefore the multiplicity of $(5, 2, 1)$ reduces to 0. So we are left with the sequence

$$(4, 3, 1).$$

By subtracting $(\rho) = (2, 1, 0)$ from this sequence we get

$$(2, 2, 1)$$

and using the map (6.10) we get that the weight associated to this sequence is λ_2. Therefore we get the fusion product

$$V^{\lambda_1+\lambda_2} \otimes_2 V^{2\lambda_1} = V^{\lambda_2}.$$

In future work we plan to investigate the dependence of the fusion coefficients on level k by using these techniques.

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