SOME APPLICATIONS OF VERTEX OPERATORS TO KAC-MOODY ALGEBRAS

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1. INTRODUCTION

This is an account of some of my recent work [2,3] which has involved applications of vertex operators to Kac-Moody Lie theory. For V the basic $A_1^{(1)}$-module in the principal realization given by Lepowsky and Wilson [13], one may use vertex operators to describe the decomposition $V \otimes V = S(V) \oplus A(V)$ of $V \otimes V$ into symmetric tensors $S(V)$ and antisymmetric tensors $A(V)$. This turns out to be precisely the decomposition of $V \otimes V$ into two "strings" of level two standard $A_1^{(1)}$-modules which I found in [1]. This result has a remarkable application to the construction of the hyperbolic algebra F with Dynkin diagram  .

In [2] Frenkel and I gave a $\mathbb{Z}$-graded construction of F such that the 0, 1 and -1 graded pieces (levels) were $A_1^{(1)}$ extended by the derivation d, V, and its contragredient module V^*, respectively. The higher levels were graded pieces of quotients of free Lie algebras by a graded ideal. For level 2 these were precisely determined to be $V \wedge V \approx A(V)$ modulo a single irreducible component (the top module of the antisymmetric string), and similarly for level -2 using V^* in place of V. This gave the first precise formula for "higher level" hyperbolic root multiplicities beyond the general formula of Moody and Berman [17]. These multiplicities have a remarkable relationship with the values of the classical partition function which has led to conjectures concerning upper bounds for all hyperbolic root multiplicities [6]. Different ways of applying vertex operators to the construction of hyperbolic algebras will be discussed by others in this workshop, but, as of this writing, none has yet explained those root multiplicities for F which are known precisely.

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† Partially supported by the National Science Foundation through the Mathematical Sciences Research Institute

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Vertex Operators in Mathematics and Physics - Proceedings of a Conference November 10-17, 1983. Publications of the Mathematical Sciences Research Institute #3, Springer-Verlag, 1984.

In order to extend the results mentioned above to the hyperbolic algebra with Dynkin diagram  I have recently studied the decomposition of $V \otimes V$ where V is the basic $A_2^{(2)}$-module. The techniques of [1,2] give this decomposition into two strings of level 2 standard modules with outer multiplicities, remarkably, equal to the coefficients of the Rogers-Ramanujan identities. From numerical data sent to me by V.G. Kac, it appears that the second level of this hyperbolic algebra consists of those irreducible components of $V \otimes V$ whose highest weights have odd principal degree > 1 relative to $1 \otimes 1$. One expects this to follow as for the algebra F from the decomposition $V \otimes V = S(V) \oplus A(V)$ and from the identification of $A(V)$ modulo one irreducible component with the second level of the $\mathbb{Z}$-graded hyperbolic algebra. The point of interest to those at this workshop is how the vertex operator techniques used to find $V \otimes V = S(V) \oplus A(V)$ in the $A_1^{(1)}$ case can be modified for the $A_2^{(2)}$ case. The proof in the $A_1^{(1)}$ case depended on the introduction of an auxiliary vertex operator on $V \otimes V$ with components which form a Clifford algebra, which commute with the action of the principal Heisenberg subalgebra on $V \otimes V$, and which anticommute with the action of the real root vectors on $V \otimes V$. In the case of $A_2^{(2)}$ the components of the analogous auxiliary vertex operator have much more complicated relations with each other and with the real root vector action on $V \otimes V$. In fact, current joint work with J. Lepowsky shows that one is dealing here with Z-algebras [15-16]. One may hope to generalize these results to all affine algebras and apply the theory of Z-algebras to the decomposition of more general tensor products.

In [3] Frenkel and I were able to construct highest weight representations for all "classical" affine algebras and superalgebras. These consist of the orthogonal series ($D_\ell^{(1)}$, $B_\ell^{(1)}$, $D_{\ell+1}^{(2)}$), the symplectic series ($C_\ell^{(1)}$, $B^{(1)}(0,\ell)$, $C^{(2)}(\ell+1)$), and the general linear series ($A_{\ell-1}^{(1)}$, $A_{2\ell-1}^{(2)}$, $A_{2\ell}^{(2)}$, $A^{(2)}(0,2\ell-1)$, $A^{(4)}(0,2\ell)$). The representations are given by certain "normally ordered" quadratic expressions whose linear factors generate an infinite-dimensional Clifford or Weyl algebra. This provides representations of the affine algebras on exterior or symmetric algebras of polynomials, respectively, in perfect analogy with the spinor (D_ℓ, B_ℓ) and oscillator (C_ℓ, $B(0,\ell)$) representations. The linear generators from the Clifford or Weyl algebra play the role in quantum mechanics of creation and annihilation operators for particles obeying Fermi or Bose statistics, respectively. We, therefore, named the corresponding constructions of affine algebras as fermionic or bosonic. The orthogonal series then has fermonic representations which are the standard representations previously called spinor representations [4,5,9]. The symplectic series has bosonic representations which are non-standard highest weight representations whose existence, at least for $C_\ell^{(1)}$, was independently noticed by H. Garland and M. Primc. For the general linear series one has both fermionic (standard) and bosonic (non-standard) representations. For the orthogonal series the fermionic (spinor) construction was shown by Frenkel [5] to be isomorphic to the vertex construction, thus interpreting the boson-fermion correspondence of physics in the framework of Kac-Moody theory. The fermions are the linear Clifford generators and the bosons are the normally ordered quadratic elements which form a Lie algebra. Both are realized as vertex operators. Changing the Clifford generators to Weyl generators gives the bosonic (oscillator) construction of the symplectic series, but one no longer has an alternative description using vertex operators. A vertex construction does exist for type A affine algebras, and for type $A_{2\ell-1}^{(2)}$ it was instrumental in discovering the way to construct the general linear series by normally ordered quadratic elements with a "twisted" generating function.

2. A TENSOR PRODUCT DECOMPOSITION AND APPLICATIONS

Let $\mathfrak{s}$ be the infinite-dimensional Heisenberg algebra with generators $h(n)$, $n \in 2\mathbb{Z}+1$, and where

(2.1) $$[h(m), h(n)] = m\delta_{m,-n}.$$

Let V be the symmetric algebra of polynomials in $\{h(-n) \mid 0 < n \in 2\mathbb{Z}+1\}$. Then V is an irreducible $\mathfrak{s}$-module. Define the vertex operator

$$X(2h,z) = -\tfrac{1}{2}\exp\Big(\sum_k \frac{z^k}{k}\,2h(-k)\Big)\exp\Big(-\sum_k \frac{z^{-k}}{k}\,2h(k)\Big) \tag{2.2}$$

where the summations are over $0 < k \in 2\mathbf{Z}+1$ (as they are throughout this section unless otherwise indicated). Let

$$X(2h,z) = \sum_{n\in\mathbf{Z}} X_n(2h)\, z^{-n} \tag{2.3}$$

define the homogeneous components $X_n(2h)$ of $X(2h,z)$. Then the operators $X_n(2h)$ for $n \in \mathbf{Z}$ are well defined on V and satisfy commutation relations with each other and with $\mathfrak{s}$ so as to provide the basic representation of the affine Kac-Moody algebra $\mathfrak{g}$ of type $A_1^{(1)}$ in its principal realization [13]. In particular, we have

$$[h(k), X_n(2h)] = 2\,X_{n+k}(2h),\ k \in 2\mathbf{Z}+1,\ n \in \mathbf{Z}, \tag{2.4}$$

$$[X_m(2h), X_n(2h)] = \begin{cases} m\delta_{m,-n} & \text{if } m,n \in 2\mathbf{Z} \\ -m\delta_{m,-n} & \text{if } m,n \in 2\mathbf{Z}+1 \\ 2h(m+n) & \text{if } m \in 2\mathbf{Z}+1,\ n \in 2\mathbf{Z}. \end{cases} \tag{2.5}$$

One way of doing such computations with vertex operators is to use contour integrals and normal ordering lemmas. For example, we define the normally ordered product of two vertex operators

$$:X(2h,z)X(2h,w): = \tfrac{1}{4}\exp\Big(\sum \frac{z^k}{k}\,2h(-k)\Big)\exp\Big(\sum \frac{w^k}{k}\,2h(-k)\Big) \cdot \exp\Big(-\sum \frac{z^{-k}}{k}\,2h(k)\Big)\exp\Big(-\sum \frac{w^{-k}}{k}\,2h(k)\Big) \tag{2.6}$$

so that all the annhilation operators are applied first, and then all the creation operators are applied. Then we have the following.

Lemma 1. For $|z| > |w|$,

$$X(2h,z)X(2h,w) = \Big(\frac{z-w}{z+w}\Big)^2 :X(2h,z)X(2h,w):.$$

__Proof.__ Let $X(2h,z) = -\frac{1}{2}\exp(A)\exp(B)$ and $X(2h,w) = -\frac{1}{2}\exp(C)\exp(D)$, then

$$[B,C] = -4\sum_k \frac{(w/z)^k}{k} = 2\log\Big(\frac{z-w}{z+w}\Big) \text{ if } |z| > |w|.$$

This scalar commutes with the operators A, B, C, D, so by the Baker-Campbell-Hausdorff formula,

$$\exp(B)\exp(C) = \exp(C)\exp(B)\exp([B,C]).$$

Let us abbreviate $\frac{1}{2\pi i}\int_C f(z)\,dz$ by just $\int_C f(z)\,dz$. Then we can compute the bracket

$$\begin{aligned} [X_m(2h), X_n(2h)] &= \Big[\int_{C_{R_1}} X(2h,z)\, z^{m-1}\,dz,\ \int_{C_{R_2}} X(2h,w)\, w^{n-1}\,dw\Big] \\ &= \int_{C_{R_1}} \Big(\int_{C_R} X(2h,z)\,X(2h,w)\, z^{m-1}\,dz - \int_{C_r} X(2h,w)\,X(2h,z)\, z^{m-1}\,dz\Big)\, w^{n-1}\,dw \\ &= \int_{C_{R_1}}\int_{C_R\setminus C_r} \Big(\frac{z-w}{z+w}\Big)^2 :X(2h,z)\,X(2h,w):\, z^{m-1}dz\, w^{n-1}dw \end{aligned} \tag{2.7}$$

where the contours are circles about the origin having radii satisfying $r < R_1 < R$. The only residue of the inner integral coming from the pole $z = -w$ is

$$\frac{d}{dz}\big((z-w)^2 :X(2h,z)\,X(2h,w):\big)_{z=-w} = (-1)^m w^m \Big(m - \sum_{k\in 2\mathbf{Z}+1} w^k\, 2h(-k)\Big) \tag{2.8}$$

so one gets

$$(2.9)\qquad \int_{C_{R_1}} (-1)^m \Big(m - \sum_{k\in 2\mathbf{Z}+1} w^k 2h(-k)\Big)\, w^{m+n-1}\, dw$$

$$= \begin{cases} (-1)^m\, m\, \delta_{m,-n} & \text{if } m+n \in 2\mathbf{Z} \\ (-1)^{m+1}\, 2h(m+n) & \text{if } m+n \in 2\mathbf{Z}+1. \end{cases}$$

Now consider the tensor product $V \otimes V$. Denote by $h^1(m)$ the action of $\mathfrak{s}$ on the first tensor factor and by $h^2(m)$ the action on the second factor. Then the action of $\mathfrak{g}$ on $V \otimes V$ is given by $h^1(m) + h^2(m)$, $m \in 2\mathbf{Z}+1$, $X_n(2h^1) + X_n(2h^2)$, $n \in \mathbf{Z}$. The central element which acted as the scalar 1 on V acts as 2 on $V \otimes V$. This is just the realization of $\mathfrak{g}$ as the diagonal subalgebra of $\mathfrak{g}^1 \times \mathfrak{g}^2$. Define the generating functions

$$(2.10)\qquad E^+(h^i, z) = \exp\Big(\sum_k \frac{z^{-k}}{k}\, h^i(k)\Big) \quad \text{and}$$
$$E^-(h^i, z) = \exp\Big(-\sum_k \frac{z^k}{k}\, h^i(-k)\Big) \quad \text{for } i = 1,2.$$

Then one has directly the auxiliary operator

$$(2.11)\qquad E^-(h^1+h^2, z)\, X(2h^1, z)\, E^+(h^1+h^2, z) = -\tfrac{1}{2}\, X(h^1-h^2, z)$$

which obviously commutes with $\mathfrak{s}$ on $V \otimes V$. To find out what relations hold among the components $X_n(h^1 - h^2)$ of $X(h^1 - h^2, z)$ we need the following.

<u>Lemma</u> 2. For $|z| > |w|$,

$$X(h^1 - h^2, z)\, X(h^1 - h^2, w) = \frac{z-w}{z+w} {:} X(h^1 - h^2, z)\, X(h^1 - h^2, w){:}.$$

<u>Proof</u>. Following the proof of Lemma 1, the only difference is that

$$[B,C] = -2 \sum_k \frac{(w/z)^k}{k} = \log\Big(\frac{z-w}{z+w}\Big).$$

The crucial difference between these operators and those before which formed a Lie algebra, is that the normal ordering factor here is antisymmetric in z and w, whereas before it was symmetric. It means that in order to obtain the inner contour integral over $C_R \setminus C_r$ we must anticommute components of the auxiliary vertex operator. One easily obtains the following Clifford algebra.

<u>Corollary</u>. For all $m, n \in \mathbf{Z}$,

$$(2.12)\qquad \{X_m(h^1 - h^2), X_n(h^1 - h^2)\} = 2(-1)^m\, \delta_{m,-n}.$$

To find the relations between components of $X(h^1 - h^2, z)$ and $X(2h^i, w)$, $i = 1,2$, we need the following.

<u>Lemma</u> 3. For $|z| > |w|$ we have

$$X(h^1 - h^2, z)\, X(2h^1, w) = \frac{z-w}{z+w} {:} X(h^1 - h^2, z)\, X(2h^1, w){:},$$
$$X(h^1 - h^2, z)\, X(2h^2, w) = \frac{z+w}{z-w} {:} X(h^1 - h^2, z)\, X(2h^2, w){:}$$

and for $|w| > |z|$ we have

$$X(2h^1, w)\, X(h^1 - h^2, z) = \frac{w-z}{w+z} {:} X(2h^1, w)\, X(h^1 - h^2, z){:},$$
$$X(2h^2, w)\, X(h^1 - h^2, z) = \frac{w+z}{w-z} {:} X(2h^2, w)\, X(h^1 - h^2, z){:}.$$

<u>Corollary</u>. For all $m, n \in \mathbf{Z}$,

$$(2.13)\qquad \{X_m(h^1 - h^2), X_n(2h^1)\} = 2\,(-1)^m\, X_{m+n}(h^1 + h^2),$$

$$(2.14)\qquad \{X_m(h^1 - h^2), X_n(2h^2)\} = 2\, X_{m+n}(h^1 + h^2)$$

so that for $m \in 2\mathbf{Z}+1$,

$$(2.15)\qquad \{X_m(h^1 - h^2), X_n(2h^1) + X_n(2h^2)\} = 0.$$

One may also see easily that

(2.16) $X_m(h^1-h^2)(1\otimes 1) = \int_C X(h^1-h^2, z)(1\otimes 1)z^{m-1}\, dz = 0$, for $m > 0$.

Now consider the collection of vectors in $V \otimes V$,

(2.17) $$\{X_{-2n_1-1}(h^1-h^2)X_{-2n_2-1}(h^1-h^2)\cdots X_{-2n_k-1}(h^1-h^2)(1\otimes 1) \mid k \geq 0,\quad n_1 > n_2 > \cdots > n_k \geq 0\}.$$

From (2.12), (2.15), (2.16) one finds these vectors are linearly independent and are killed by $X_n(2h^1) + X_n(2h^2)$ for $0 < n \in \mathbf{Z}$, and by $h^1(k) + h^2(k)$ for $0 < k \in 2\mathbf{Z}+1$ (which represent positive root vectors of $\mathfrak{g}$ which kill $1 \otimes 1$). The principally specialized character of the space of highest weight vectors having basis (2.17) is

(2.18) $$e^{\lambda} \prod_{n\geq 1} (1 + u^{2n-1})$$

where λ is the weight of $1 \otimes 1$. But the decomposition of $V \otimes V$ into two "strings" of level 2 irreducible $\mathfrak{g}$-modules is known [1] to be

(2.19) $$V \otimes V = \sum_{m\geq 0} (a_m V^{2\omega_2+m\omega_3} + b_m V^{2\omega_1+(m+1)\omega_3})$$

where the outer multiplicities a_m, b_m are defined by

(2.20) $$\sum_{m\geq 0} (a_m x^{2m} + b_m x^{2m+1}) = \prod_{j\geq 1} (1 + x^{2j-1}).$$

The fundamental weights of $\mathfrak{g}$ are ω_1 and ω_2, dual to the simple roots α_1 and α_2, $V = V^{\omega_2}$, and $\omega_3 = -\alpha_1-\alpha_2$. This implies that the principally specialized character of the space Ω of highest weight vectors in $V \otimes V$ is exactly (2.18), so that (2.17) is a basis of Ω.

Since $X_m(h^1 - h^2) = (-1)^m X_m(h^2 - h^1)$ it is clear that the typical vector in (2.17) is symmetric if k is even and antisymmetric if k is odd. To determine which string that vector falls into, note that $1 \otimes 1$ has weight $\lambda = 2\omega_2$, $2\omega_1 = 2\omega_2 + \alpha_1$ and ω_3 has even

principal degree. So when k is even we get a vector of weight $2\omega_2 + m\omega_3$ and when k is odd we get a vector of weight $2\omega_1 + (m+1)\omega_3$ for some $m \geq 0$. This gives the following.

Theorem 1 [2]. We have

$$S(V) = \sum_{m\geq 0} a_m V^{2\omega_2+m\omega_3}$$

$$A(V) = \sum_{m\geq 0} b_m V^{2\omega_1+(m+1)\omega_3}.$$

This result has a remarkable application to the determination of certain root multiplicities in the hyperbolic algebra $\mathcal{F}$ with Dynkin diagram  . This algebra has a $\mathbf{Z}$-graded construction [2]

(2.21) $$\mathcal{F} = \sum_{n\in\mathbf{Z}} \mathcal{F}_n$$

such that $\mathcal{F}_0 = \mathfrak{g} + \mathbb{C}d$ is the usual extension of $\mathfrak{g}$ by the derivation d, $\mathcal{F}_1 = V^{\omega_2-\omega_3} \approx V$ and $\mathcal{F}_{-1} = V^{-\omega_2+\omega_3} \approx V^*$ is the dual (contragredient) $\mathfrak{g}$-module. The higher "levels" of $\mathcal{F}$ are much more complicated, being the graded pieces of free Lie algebras generated by $\mathcal{F}_1$ (for $n > 0$) or $\mathcal{F}_{-1}$ (for $n < 0$) modulo a graded ideal

(2.22) $$I = \sum_{n\in\mathbf{Z}} I_n.$$

It turns out that

(2.23) $$\mathcal{F}_{\pm 2} \approx (\mathcal{F}_{\pm 1} \wedge \mathcal{F}_{\pm 1}) / I_{\pm 2}$$

can be precisely determined using Theorem 1 and its analog for $V^* \otimes V^* = S(V^*) \oplus A(V^*)$. One finds that

(2.24) $$\mathcal{F}_1 \wedge \mathcal{F}_1 = A(V^{\omega_2-\omega_3}) = \sum_{m\geq 0} b_m V^{2\omega_1+(m-1)\omega_3}$$

and

$$I_2 = V^{2\omega_1-\omega_3}. \tag{2.25}$$

This gives a precise formula for the hyperbolic root multiplicities on the second level of F which shows them to be closely related to the values of the classical partition function. Further details may be found in [2].

3. ANOTHER TENSOR PRODUCT DECOMPOSITION

Joint work is in progress with J. Lepowsky on the analogue of the results from section 2 in the case when $\mathfrak{g}$ is of type $A_2^{(2)}$. The decomposition $V \otimes V = S(V) \oplus A(V)$ for the basic $A_2^{(2)}$-module $V = V^{\omega_2}$ may be applied to the hyperbolic algebra $\bar{A}_2^{(2)}$ with Dynkin diagram  , simple roots $\alpha_1, \alpha_2, \alpha_3$ and fundamental weights $\omega_1, \omega_2, \omega_3$.

<u>Theorem</u> 2. We have the decomposition

$$V \otimes V = \sum_{m\geq 0} (a_m V^{2\omega_2+2m\omega_3} + b_m V^{\omega_1+(2m+1)\omega_3}) \tag{3.1}$$

into irreducible level 2 $\mathfrak{g}$-modules, where the outer multiplicities are given by

$$\sum_{m\geq 0} a_m x^m = \prod_{n\geq 1} (1 - x^{5n-1})^{-1} (1 - x^{5n-4})^{-1} \tag{3.2}$$

$$\sum_{m\geq 0} b_m x^m = \prod_{n\geq 1} (1 - x^{5n-2})^{-1} (1 - x^{5n-3})^{-1}. \tag{3.3}$$

These are the product sides of the famous Rogers-Ramanujan identities [10, 14-16], which provide two combinatorial descriptions of the coefficients. We have that a_m equals the number of partitions of m into parts $\equiv$ 1,4 mod 5 which equals the number of partitions of m into parts with difference at least 2. Also, b_m equals the number of partitions of m into parts $\equiv$ 2, 3 mod 5 which equals the number of partitions of m into parts with difference at least 2 and no part less

than 2.

The principally specialized character of the space Ω of highest weight vectors in $V \otimes V$ is

$$ch(\Omega) = e^{2\omega_2} \sum_{m\geq 0} (a_m u^{3m} + b_m u^{3m+1}). \tag{3.4}$$

Using some auxiliary vertex operator one expects to find a basis for Ω which explains the combinatorial descriptions of a_m and b_m. In fact, numerical data on the root muliplicities of the hyperbolic algebra $\bar{A}_2^{(2)}$ provided by V.G. Kac indicate that the second level of that algebra consists of those irreducible components of $V \otimes V$ having highest weight vector of odd principal degree greater than 1 relative to $1 \otimes 1$. This indicates that A(V) consists of all components having highest weight vector of odd principal degree and that $I_2 \approx V^{\omega_1+\omega_3}$. Instead of two strings, one symmetric and one antisymmetric, here we have the following.

<u>Theorem</u> 3. If V is the basic $A_2^{(2)}$-module and $V \otimes V = S(V) \oplus A(V)$ then

$$S(V) = \sum_{m\geq 0} (a_{2m} V^{2\omega_2+4m\omega_3} + b_{2m+1} V^{\omega_1+(4m+3)\omega_3}) \tag{3.5}$$

$$A(V) = \sum_{m\geq 0} (a_{2m+1} V^{2\omega_2+(4m+2)\omega_3} + b_{2m} V^{\omega_1+(4m+1)\omega_3}). \tag{3.6}$$

The basic module for $A_2^{(2)}$ is constructed as follows [8]. Let $\mathfrak{s}$ be the Heisenberg algebra with generators h(n), $n \equiv \pm 1 \bmod 6$ such that

$$[h(m), h(n)] = m\, \delta_{m,-n}. \tag{3.7}$$

V is the symmetric algebra of polynomials in $\{h(-n) \mid 0 < n \in \mathbf{Z}, n \equiv \pm 1 \bmod 6\}$. Then V is an irreducible $\mathfrak{s}$-module. Let ϵ be a primitive 6^{th} root of unity. Define the vertex operator

$$(3.8)\quad X(2h,z) = \exp\Big(\sum_k \frac{z^k}{k}\,\frac{\epsilon^k+1}{2^{1/2}}\,2h(-k)\Big)\exp\Big(-\sum_k \frac{z^{-k}}{k}\,\frac{\epsilon^{-k}+1}{2^{1/2}}\,2h(k)\Big)$$

where the summations are over $0 < k \in \mathbf{Z}$, $k \equiv \pm1 \bmod 6$ (as they are throughout this section unless otherwise indicated). Then $\mathfrak{s}$ and the components $X_n(2h)$, $n \in \mathbf{Z}$, of $X(2h,z)$ provide the basic representation of $A_2^{(2)}$ on V in the principal realization. To find the bracket of two vertex components, for example, one uses the following.

<u>Lemma</u> 4. For $|z| > |w|$,

$$X(2h,z)\,X(2h,w) = \Big(\frac{z-w}{z+w}\Big)^2\Big(\frac{z-\epsilon w}{z+\epsilon w}\Big)\Big(\frac{z+\epsilon^2 w}{z-\epsilon^2 w}\Big) : X(2h,z)\,X(2h,w):.$$

The representation of $\mathfrak{g}$ on $V \otimes V$ is given by $h^1(m) + h^2(m)$, $m \equiv \pm1 \bmod 6$ and $X_n(2h^1) + X_n(2h^2)$, $n \in \mathbf{Z}$. Defining generating functions

$$(3.9)\qquad E^+(h^i,z) = \exp\Big(\sum_k \frac{z^{-k}}{k}\,\frac{\epsilon^{-k}+1}{2^{1/2}}\,h^i(k)\Big)$$

$$(3.10)\qquad E^-(h^i,z) = \exp\Big(-\sum_k \frac{z^{k}}{k}\,\frac{\epsilon^{k}+1}{2^{1/2}}\,h^i(-k)\Big)$$

for $i=1,2$ one finds the auxiliary operator

$$(3.11)\qquad E^-(h^1+h^2,z)\,X(2h^1,z)\,E^+(h^1+h^2,z) = X(h^1-h^2,z).$$

The components $X_n(h^1 - h^2)$ obviously commute with $\mathfrak{s}$ on $V \otimes V$. We need the following.

<u>Lemma</u> 5. For $|z| > |w|$,

$$X(h^1-h^2,z)\,X(h^1-h^2,w) =$$

$$\Big(\frac{z-w}{z+w}\Big)\Big(\frac{z-\epsilon w}{z+\epsilon w}\Big)^{1/2}\Big(\frac{z+\epsilon^2 w}{z-\epsilon^2 w}\Big)^{1/2} : X(h^1-h^2,z)X(h^1-h^2,w):.$$

At this point the occurence of the fractional exponents makes the use of the contour integral technique much more difficult. However, the alternative technique of formal variables and "correction factors" [10, 12, 15-16] easily shows that the components of this auxiliary vertex operator satisfy so-called "generalized anticommutation relations" which give them the structure of a "Z-algebra". While much more complicated than (2.12), this structure does allow one to determine a set of monomials

$$(3.12)\qquad X_{-m_1}(h^1-h^2)\,X_{-m_2}(h^1-h^2)\cdots X_{-m_n}(h^1-h^2)(1\otimes 1)$$

which are a basis for the "vacuum space" of $\mathfrak{s}$ (vectors killed by $h^1(m) + h^2(m)$ for $0 < m \equiv \pm1 \bmod 6$). It then remains to find the subspace of highest weight vectors also killed by $X_n(2h^1) + X_n(2h^2)$ for all $0 < n \in \mathbf{Z}$. That also requires the techniques of Z-algebras. Details will appear elsewhere.

4. THE VERTEX REPRESENTATION OF $\mathfrak{gl}^{(2)}(2\ell)$

In [3] constructions were given of all the "classical" affine algebras; the orthogonal series $D_\ell^{(1)}$, $B_\ell^{(1)}$, $D_{\ell+1}^{(2)}$, the symplectic series $C_\ell^{(1)}$, $B^{(1)}(0,\ell)$, $C^{(2)}(\ell+1)$, and the general linear series $A_{\ell-1}^{(1)}$, $A_{2\ell-1}^{(2)}$, $A_{2\ell}^{(2)}$, $A^{(2)}(0,2\ell-1)$, $A^{(4)}(0,2\ell)$. The representations of the orthogonal and symplectic series are quite analogous to the well-known spinor and oscillator representations, respectively, in the finite-dimensional theory. The representations are given by certain "normally ordered" quadratic expressions whose linear factors come from an infinite-dimensional Clifford or Weyl algebra. As was explained in the introduction, these are called fermonic or bosonic constructions, respectively. The fermionic constructions of the orthogonal series were previously known [4,5,9], as was the existence of a bosonic construction of $C_\ell^{(1)}$ [Garland and, independently, Primc, unpublished]. Essentially new was the possibility of having both fermionic and bosonic constructions of the general linear series. The homogeneous vertex representation [5,7] of type $A^{(2)}_{2\ell-1}$ (or more precisely, of $\mathfrak{gl}^{(2)}(2\ell)$) was essential to the discovery of these constructions. I will describe how this was done.

Consider the Lie algebra involution σ of $\mathfrak{gl}(2\ell)$ given by

$$(4.1)\qquad \sigma\begin{bmatrix} m & n \\ p & q \end{bmatrix} = \begin{bmatrix} -q^t & n^t \\ p^t & -m^t \end{bmatrix}.$$

Then $\mathfrak{gl}(2\ell) = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0 \approx \mathfrak{sp}(2\ell)$ is the fixed point set of σ and on $\mathfrak{g}_1$, σ acts as -1. Then we have

$$(4.2)\qquad \mathfrak{gl}^{(2)}(2\ell) = \mathfrak{g}_0 \otimes \mathbb{C}[t^2, t^{-2}] + \mathfrak{g}_1 \otimes t\mathbb{C}[t^2, t^{-2}] + \mathbb{C}c$$

where c is central and

$$(4.3)\qquad [x \otimes t^i, y \otimes t^j] = [x,y] \otimes t^{i+j} + i\, \delta_{i,-j} \langle x,y\rangle c$$

with invariant bilinear form $\langle x,y\rangle = \mathrm{Tr}(xy)$ on $\mathfrak{gl}(2\ell)$. This is a subalgebra of

$$(4.4)\qquad \mathfrak{gl}^{(1)}(2\ell) = \mathfrak{gl}(2\ell) \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c.$$

One usually extends these algebras by adjoining the derivation $d = t\frac{d}{dt}$,

$$(4.5)\qquad \hat{\mathfrak{gl}}^{(i)}(2\ell) = \mathfrak{gl}^{(i)}(2\ell) + \mathbb{C}d, \quad i=1,2.$$

Let E_{ij}, $1 \le i, j \le 2\ell$ be the standard basis of $\mathfrak{gl}(2\ell)$, then $h_i = E_{ii}$, $1 \le i \le 2\ell$, is an orthonormal basis for the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(2\ell)$. Let $\mathfrak{h}_0$ be the Cartan subalgebra of $\mathfrak{g}_0$ with basis $\bar{h}_i = h_i - h_{i+\ell}$, $1 \le i \le \ell$, and let $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ with $\mathfrak{h}_1 \subset \mathfrak{g}_1$. Using the form $\langle h_i, h_j\rangle = \delta_{i,j}$ to identify the dual space $\mathfrak{h}^*$ with $\mathfrak{h}$ we find the root system of $\mathfrak{gl}(2\ell)$ is

$$(4.6)\qquad \Phi = \{h_i - h_j \mid 1 \le i \ne j \le 2\ell\}.$$

For $x \in \mathfrak{gl}(2\ell)$ let $x^0 = \frac{1}{2}(x + \sigma x)$ and $x^1 = \frac{1}{2}(x - \sigma x)$ denote the projections of x onto $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively. Then $\Phi^0 \subset \mathfrak{h}_0$ is the root system for $\mathfrak{g}_0$ of type C_ℓ. With respect to $\mathfrak{h}_0$ the root system of $\mathfrak{g}_1$ consists of the short roots of Φ^0.

For $x \in \mathfrak{gl}(2\ell)$, $n \in \mathbb{Z}$, let us denote

$$(4.7)\qquad x(n/2) = \begin{cases} x^0 \otimes t^n & \text{if } n \in 2\mathbb{Z} \\ x^1 \otimes t^n & \text{if } n \in 2\mathbb{Z}+1 \end{cases}$$

in $\mathfrak{gl}^{(2)}(2\ell)$. To get the homogeneous vertex representation of $\mathfrak{gl}^{(2)}(2\ell)$ we begin with the Heisenberg algebra $\mathfrak{h}_\sigma$ generated by

$$(4.8)\qquad \{h_i(m),\ c \mid 1 \le i \le 2\ell,\ m \in \tfrac{1}{2}\mathbb{Z}\}$$

where

$$(4.9)\qquad [h_i(m), h_j(n)] = \begin{cases} m\,\delta_{m,-n}\langle h_i^0, h_j^0\rangle c & \text{if } m,n \in \mathbb{Z} \\ m\,\delta_{m,-n}\langle h_i', h_j'\rangle c. & \text{if } m,n \in \mathbb{Z}+\frac{1}{2} \end{cases}$$

Let $S(\mathfrak{h}_\sigma^-)$ denote the symmetric algebra of polynomials in $\{h_i(-m) \mid 1 \le i \le 2\ell,\ 0 < m \in \frac{1}{2}\mathbb{Z}\}$. Let $P = \sum_{i=1}^{2\ell} \mathbb{Z}h_i$ denote the weight lattice of $\mathfrak{gl}(2\ell)$ which projects to P^0, the weight lattice of $\mathfrak{g}_0$.

We have the root lattice $Q = \sum_{i=1}^{2\ell-1} \mathbb{Z}\alpha_i$ of $\mathfrak{gl}(2\ell)$ (where $\alpha_i = h_i - h_{i+1}$, $1 \le i \le 2\ell-1$, are the simple roots) which projects onto the root lattice Q^0 of $\mathfrak{g}_0$. There are two cosets of Q^0 in P^0, $P^0 = Q^0 \cup Q^1$. Let $\mathbb{C}[P^0]$ denote the group algebra generated by formal exponentials e^μ, $\mu \in P^0$. Then one has the two representation spaces

$$(4.10)\qquad V_k^\sigma = S(\mathfrak{h}_\sigma^-) \otimes \mathbb{C}[Q^k],\ k=0,1$$

where the action of $h(m)$, $m \neq 0$ is as usual on $S(h_\sigma^-)$, but

$$h(0)(v \otimes e^{\mu}) = \langle h, \mu\rangle (v \otimes e^{\mu}). \tag{4.11}$$

The rest of $gl^{(2)}(2\ell)$ is represented by the action of the homogeneous components of the vertex operators

$$X^{\sigma\epsilon}(\mu,z) = \tag{4.12}$$

$$\exp\Big(2 \sum_{n\geq 1} \frac{z^n}{n} \mu(-n/2)\Big)\exp(2 \log(z)\mu(0)+\mu^0)$$

$$\cdot \exp\Big(-2 \sum_{n\geq 1} \frac{z^{-n}}{n} \mu(n/2)\Big)\epsilon^0_{\mu^0}$$

for $\mu \in \Phi$. Here ϵ^0: $P^0 \times P^0 \longrightarrow \{\pm 1\}$ is a bilinear function which must satisfy certain properties discussed below, and $\epsilon^0_{\mu^0}$ acts by

$$\epsilon^0_{\mu^0}(v \otimes e^{\lambda}) = \epsilon^0(\mu^0, \lambda)(v \otimes e^{\lambda}). \tag{4.13}$$

The components of $X^{\sigma\epsilon}(\mu,z)$ for $\mu \in P$ are defined by

$$X^{\sigma\epsilon}(\mu,z) = \sum_{m\in Z} X^{\sigma\epsilon}_m(\mu)\, z^{-m} \tag{4.14}$$

where $Z = \mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ depending on whether μ^0 is in Q^0 or Q^1. This is determined by the action of the middle exponential of (4.12) in $\mathbb{C}[Q^k]$,

$$\exp(2 \log(z)\mu(0)+\mu^0)e^{\lambda} = z^{2\langle\mu^0,\lambda^0\rangle+\langle\mu^0,\mu^0\rangle}\, e^{\lambda+\mu^0}. \tag{4.15}$$

Since $\Phi \subset Q$ the components representing $gl^{(2)}(2\ell)$ are all integral. In fact, V_0^σ is the basic module.

To compute brackets of vertex operator components one uses the following.

<u>Lemma</u> 6. For $|z| > |w|$,

$$X^{\sigma\epsilon}(\lambda,z)\, X^{\sigma\epsilon}(\mu,w) =$$

$$\frac{\epsilon^0(\lambda^0,\mu^0)(z-w)^{\langle\lambda,\mu\rangle}(z+w)^{\langle\lambda,\sigma\mu\rangle}}{(zw)^{\langle\lambda^0,\mu^0\rangle}}$$

$$:X^{\sigma\epsilon}(\lambda,z)X^{\sigma\epsilon}(\mu,w):.$$

In order to find commutation relations one must have

$$\epsilon^0(\mu^0,\lambda^0)(-1)^{\langle\lambda,\mu\rangle} = \epsilon^0(\lambda^0,\mu^0) \tag{4.16}$$

for $\lambda, \mu \in \Phi$. Using the bilinearity of ϵ^0 it suffices to have condition (4.16) for the simple roots in Φ. This is easily obtained by setting $\epsilon^0(\alpha_i^0, \alpha_j^0) = 1$ for $i \leq j$, so the other values are determined by (4.16).

We wish to understand how to write

$$X^{\sigma\epsilon}(h_i - h_j, z) =: X^{\sigma\epsilon}(h_i, z)\, X^{\sigma\epsilon}(-h_j, z): \tag{4.17}$$

so that the components of $X^{\sigma\epsilon}(\pm h^i, z)$ form a Clifford algebra. It is necessary to extend the definition of ϵ^0 to ϵ: $P \times P^0 \rightarrow \{\pm 1\}$ so as to obtain

$$\epsilon(\lambda,\mu^0) = -(-1)^{\langle\lambda,\mu\rangle}\, \epsilon(\mu,\lambda^0). \tag{4.18}$$

Define for $1 \leq i, j \leq \ell$,

$$\epsilon(h_i, h_j^0) = \begin{cases} +1 & \text{if } i \leq j \\ -1 & \text{if } i > j \end{cases} \tag{4.19}$$

and

$$\epsilon(h_{i+\ell}, h_j^0)(-1)^{2\langle h_i^0, h_j^0\rangle} = \epsilon(h_i, h_j^0) \tag{4.20}$$

so that

$$(4.21)\qquad \epsilon(h_i, h_j^0) = \begin{cases} \epsilon(h_{i+\ell}, h_j^0) & \text{if } i \neq j \\ -\epsilon(h_{i+\ell}, h_j^0) & \text{if } i = j. \end{cases}$$

Using the bilinear function ϵ determined by these conditions in place of ϵ^0 in (4.12) we get anticommutation relations among the components $X_m^{\sigma\epsilon}(\pm h_i)$, for which $m \in \mathbf{Z} + \frac{1}{2}$, and we get the same commutation relations as before among the components $X_n^{\sigma\epsilon}(h_i - h_j)$, $n \in \mathbf{Z}$. We find that for $m, n \in \mathbf{Z} + \frac{1}{2}$

$$(4.22)\qquad [X_m^{\sigma\epsilon}(h_i), X_n^{\sigma\epsilon}(-h_j)] = \epsilon(h_i, -h_j^0)\, \delta_{i,j}\, \delta_{m,-n}$$

for $1 \leqslant i, j \leqslant 2\ell$, and

$$(4.23)\qquad [X_m^{\sigma\epsilon}(h_i), X_n^{\sigma\epsilon}(h_{j+\ell})] = \epsilon(h_i, -h_j^0)\, (-1)^{m-\frac{1}{2}}\, \delta_{i,j}\, \delta_{m,-n}$$

for $1 \leqslant i, j \leqslant \ell$.

Let us introduce the notation

$$(4.24)\qquad a_i(m) = X_m^{\sigma\epsilon}(h_i), \quad a_i^*(m) = X_m^{\sigma\epsilon}(-h_i)$$

for $1 \leqslant i \leqslant 2\ell$, $m \in \mathbf{Z}+\frac{1}{2}$. Then we have

$$(4.25)\qquad [a_i(m), a_j^*(n)] = \delta_{i,j}\, \delta_{m,-n}, \quad 1 \leqslant i, j \leqslant 2\ell$$

$$(4.26)\qquad [a_i(m), a_{j+\ell}(n)] = (-1)^{m-\frac{1}{2}}\, \delta_{i,j}\, \delta_{m,-n}, \quad 1 \leqslant i, j \leqslant \ell$$

$$(4.27)\qquad [a_i^*(m), a_{j+\ell}^*(n)] = (-1)^{m-\frac{1}{2}}\, \delta_{i,j}\, \delta_{m,-n}, \quad 1 \leqslant i, j \leqslant \ell.$$

As in the spinor construction of the orthogonal series, one would like to consider the normally ordered quadratic expressions

$$(4.28)\qquad \sum_{k \in \mathbf{Z}+\frac{1}{2}} : a_i(m-k) a_j^*(k) :$$

for $m \in \mathbf{Z}$, as components of generating functions: $a_i(z) a_j^*(z)$: in fermonic normal order. However, there are relations among the generators

$$(4.29)\qquad a_i(z) = X^{\sigma\epsilon}(h_i, z), \quad a_i^*(z) = X^{\sigma\epsilon}(-h_i, z)$$

which can only be seen from the definitions of these vertex operators.

<u>Lemma</u> 7. For $1 \leqslant i \leqslant \ell$,

$$(4.30)\qquad X^{\sigma\epsilon}(h_{i+\ell}, z) = (-1)^{-\frac{1}{2}}\, X^{\sigma\epsilon}(-h_i, -z)$$

and

$$(4.31)\qquad X^{\sigma\epsilon}(-h_{i+\ell}, z) = (-1)^{-\frac{1}{2}}\, X^{\sigma\epsilon}(h_i, -z).$$

<u>Proof</u>. Writing out the definitions of $X^{\sigma\epsilon}(h_{i+\ell}, z)$ and $X^{\sigma\epsilon}(-h_i, -z)$ using (4.1) and (4.7) one sees that the first and third exponentials are equal. Consider the difference between the actions of their middle exponentials and their cocycles ϵ on e^{λ^0}. From (4.15) the first gives

$$z^{2\langle -h_i^0, \lambda^0\rangle + \langle h_i^0, h_i^0\rangle}\, \epsilon(h_{i+\ell}, \lambda^0)\, e^{\lambda^0 - h_i^0}$$

because $h_{i+\ell}^0 = -h_i^0$, and the second gives

$$(-z)^{2\langle -h_i^0, \lambda^0\rangle + \langle h_i^0, h_i^0\rangle}\, \epsilon(-h_i^0, \lambda^0)\, e^{\lambda^0 - h_i^0}.$$

But then from (4.20)

$$(-1)^{2\langle -h_i^0,\, \lambda^0\rangle} \epsilon(-h_i^0, \lambda^0) = \epsilon(h_{i+\ell}, \lambda^0),$$

and $\langle h_i^0, h_i^0\rangle = \frac{1}{2}$, so we get (4.30). The case of (4.31) is similar.

This shows that for $1 \leqslant i \leqslant \ell$,

$$a_{i+\ell}(z) = (-1)^{-\frac{1}{2}} a_i^*(-z) \quad \text{and}$$

$$a_{i+\ell}^*(z) = (-1)^{-\frac{1}{2}} a_i(-z), \tag{4.32}$$

so that $gl^{(2)}(2\ell)$ can be generated by the components of

$$:a_i(z)a_j^*(z):,\quad :a_i(z)a_j(-z):,\quad :a_i^*(-z)\, a_j^*(z): \text{ for } 1 \leqslant i,j \leqslant \ell. \tag{4.33}$$

Note how (4.32) gives (4.26) and (4.27) from (4.25).

This was the method by which the "twisted" constructions of the general linear series were found. It is remarkable that when the fermionic generators (4.25) are replaced by bosonic ones, or when the index set Z is changed from $\mathbf{Z}+\frac{1}{2}$ to $\mathbf{Z}$, the functions (4.33) still provide a representation of the same algebra. For further details see [3].

REFERENCES

[1] A.J. Feingold, Tensor products of certain modules for the generalized Cartan matrix Lie algebra $A_1^{(1)}$, Communications in Algebra, Vol. 9, No. 12 (1981), 1323-1341.

[2] A.J. Feingold, I. Frenkel, A hyperbolic Kac-Moody algebra and the theory of Siegel modular forms of genus 2, Math. Ann. 263 (1983), 87-144.

[3] A.J. Feingold, I. Frenkel, Classical affine algebras, Advances in Math, to appear.

[4] I. Frenkel, Spinor representations of affine Lie algebras, Proc. Natl. Acad. Sci. USA, Vol. 77, No. 11, (1980), 6303-6306.

[5] I. Frenkel, Two constructions of affine Lie algebra representations and boson-fermion correspondence in quantum field theory, J. of Funct. Anal., Vol. 44, No. 3 (1981), 259-327.

[6] I. Frenkel, Representations of Kac-Moody algebras and dual resonance models, Proceedings of the AMS-SIAM 1982 Summer Seminar on Applications of Group Theory in Physics and Mathematical Physics, July, 1982.

[7] I. Frenkel, V.G. Kac, Basic representations of affine Lie algebras and dual resonance models, Invent. Math. 62 (1980), 23-66.

[8] V.G. Kac, D.A. Kazhdan, J. Lepowsky, R.L. Wilson, Realization of the basic representations of the Euclidean Lie algebras, Advances in Math. 42, (1981), 83-112.

[9] V.G. Kac, D.H. Peterson, Spin and wedge representations of infinite-dimensional Lie algebras and groups, Proc. Natl. Acad. Sci. USA, Vol. 78, No. 6 (1981), 3308-3312.

[10] J. Lepowsky, Affine Lie algebras and combinatorial identities, Proc. 1981 Rutgers Conference on Lie algebras and Related Topics, Springer-Verlag Lecture Notes in Math. 933 (1982), 130-156.

[11] J. Lepowsky, Some constructions of the affine Lie algebra $A_1^{(1)}$, Proceedings of the AMS-SIAM 1982 Summer Seminar on Applications of Group Theory in Physics and Mathematical Physics, July, 1982.

[12] J. Lepowsky, M. Primc, Standard modules for type one affine Lie algebras, Number Theory, New York, 1982, Springer-Verlag Lecture Notes in Mathematics 1052 (1984) 194-251.

[13] J. Lepowsky, R.L. Wilson, Construction of the affine Lie algebra $A_1^{(1)}$, Comm. in Math. Phys. 62 (1978), 43-53.

[14] J. Lepowsky, R.L. Wilson, A Lie theoretic interpretation and proof of the Rogers-Ramanujan identities, Advances in Math. 45 (1982), 21-72.

[15] J. Lepowsky, R.L. Wilson, A new family of algebras underlying the Rogers-Ramanujan identities and generalizations, Proc. Natl. Acad. Sci. USA, Vol. 78 (1981), 7254-7258.

[16] J. Lepowsky, R.L. Wilson, The structure of standard modules, I: Universal algebras and the Rogers-Ramanujan identities, preprint, 1982, Invent. Math., (1984).

[17] R.V. Moody, S. Berman, Lie algebra multiplicities, Proceedings of the American Mathematical Society 76 (1979), 223-228.

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