MATH. 341. Final Examination December 15, 2000.

1. (10 points) For testing the reliability of an insertion machine, $n = 2500$ insertions were made. Let X denote the number of errors in the sample. It is assumed X has a Binomial distribution with parameter (probability of error) θ. One would like to test the hypothesis $H_0 : \theta \leq .001$ against the alternative $H_1 : \theta > .001$

(a) Compute the P-value of the test if $X = 4$.

(b) Would you reject the null hypothesis?

Solution:

(a) The p–value is

$$P(X \geq 4) = 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3)$$

$$= 1 - \binom{2500}{0} \cdot (.001)^0 \cdot (.999)^{2500} - \binom{2500}{1} \cdot (.001)^1 \cdot (.999)^{2499}$$

$$-\binom{2500}{2} \cdot (.001)^2 \cdot (.999)^{2498} - \binom{2500}{3} \cdot (.001)^3 \cdot (.999)^{2497}$$

$$= 1 - .0820 - 0.2052 - 2566 - .2139 = .2423.$$

(b) No, we do not reject H_o.

2. (15 points) In a random sample of $n = 25$ units, the sample mean is 3.25. It is assumed that the observations follow a Poisson distribution with parameter λ.

(a) Compute the moment equation estimate of λ.

(b) What is the variance of the sample mean?

(c) Estimate the standard deviation of the sample mean.

Solution:

(a) To find the method of the moments estimator, we solve for λ in $\bar{x} = \lambda$. We get $\hat{\lambda} = \bar{x} = 3.25$.

(b) The variance of $\bar{x}$ is $\mathrm{Var}(\bar{x}) = \frac{\lambda}{n} = \frac{\lambda}{25}$.

(c) An estimate of the standard deviation is $\sqrt{\frac{3.25}{25}} = 0.3605$.

2. (10 points) In a sample of $n = 200$ resistors we find $X = 22$ defective ones. Determine a confidence interval, at level of confidence $1 - \alpha = .95$, for the probability p of a defective resistor.

Solution:

The 95% confidence interval for p is

$$\hat{p} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.11 \pm 1.96 \cdot \sqrt{\frac{0.11 \cdot .89}{200}} = 0.11 \pm 0.0434 = (0.0666, 0.1534)$$

4. (15 points) A random variable X has a binomial distribution with parameters $n = 1000$ and $p = 0.25$ approximate the probability of the event $\{X < 253\}$.

Solution: We have that

$$E[X] = np = 100 \cdot (.25) = 250$$

and

$$\mathrm{Var}(X) = np(1-p) = 100 \cdot (.25)(1 - .25) \cdot = 187.5$$

By continuity correction and the central limit theorem

$$P(X < 253) \approx P\left(N(0,1) \leq \frac{252.5 - 250}{\sqrt{287.5}}\right) = \Phi(0.1826)$$

$$= (1 - 0.826) \cdot \Phi(0.1) + 0.826 \cdot \Phi(0.2) = 0.5724.$$

5. (10 points) A concrete cube has a compressive-strength X which has a log-normal distribution $\mathrm{LN}(\mu, \sigma)$ with parameters $\mu = 4$ and $\sigma = 2$.

a.) What is the probability that a concrete cube will have compressive-strength greater than 300 pounds/inch2?

b.) What is the probability in a random sample of $n = 10$ concrete cubes, there will be more than 3 cubes with compressive-strength greater than 300 pounds/inch^2?

Solution:

(a)

$$P(X \geq 300) = P(\ln(X) \geq \ln(300)) = P\left(N(0,1) \geq \frac{\ln(300) - 4}{2}\right) = 1 - \Phi(0.8519).$$

By interpolation,

$$\Phi(0.8519) = (1 - 0.519) \cdot \Phi(0.8) + (0.519) \cdot)\Phi(0.9) = 0.8026.$$

So, $P(X \geq 300) = 1 - 0.8026 = 0.1974$.

(b) Let N be the number of cubes with compressive-strength greater than 300 pounds/inch^2. Then, N is a binomial random variable with $n = 10$ and $p = 0.1974$. So,

$$P(N \geq 4) = 1 - P(N = 0) - P(N = 1) - P(N = 2) - P(N = 3)$$

$$= 1 - \binom{10}{0} \cdot (.1974)^0 \cdot (.8026)^{10} - \binom{10}{1} \cdot (.1974)^1 \cdot (.8026)^9$$
$$-\binom{10}{2} \cdot (.1974)^2 \cdot (.8026)^8 - \binom{10}{3} \cdot (.1974)^3 \cdot (.8026)^7$$
$$= 1 - .1109 - 0.2728 - .3019 - .198 = .1164.$$

6. (20 points) A random sample of size 100 is taken from a normal distribution, which has an unknown mean μ and known variance $\sigma^2 = 36.25$. The hypothesis $H_0 : \mu = 20.5$ is to be tested against the one sided alternative $H_a : \mu > 20$. The observed value of the sample mean X is 21.75. Calculate the P-value of the test. Use the attached tables of the normal distribution to give a numerical answer.

Solution: $z = \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} = \frac{\sqrt{100}(21.75-20.5)}{\sqrt{36.25}} = 2.0762$. So, the p–value is $P(N(0,1) \geq 2.0762) = 1 - \Phi(2.0762)$. By interpolation

$$\Phi(2.0762) = (1 - 0.762) \cdot \Phi(2.0) + 0.762 \cdot \Phi(2.1) = .9840.$$

So, the p–value is $1 - .9840 = 0.0160$.

7. (20 points) A random sample of $n = 50$ steel rods is taken. We wish to perform a one sample t-test of the hypothesis that the mean $\mu = 19.0$ vs the alternative hypothesis that the μ is smaller than 19.0. The sample mean is 18.65. Also, the sample standard deviation $s = 1.12$. Answer the following questions:

(a) (9 points) Calculate the t value for the test.

(b) (3 points) The number of degrees of freedom for this test is?

(c) (4 points) Is the null hypothesis rejected at the 5% level?

(d) (4 points) Is the null hypothesis rejected at the 1% level?

Solution:

(a) The t statistic is $t = \frac{\sqrt{n}(\bar{x}-\mu_0)}{s} = \frac{\sqrt{50}(18.65-19)}{1.12} = -2.2097$

(b) The degrees of freedom are $n - 1 = 49$.

(c) We reject the null hypothesis if $t \leq z_\alpha$. Since the degrees of freedom are large we use the normal table. $z_{0.05} = -1.645$. So, we reject H_0 at the 5% level.

(d) $z_{0.01} = -3.26$. So, we do not reject H_0 at the 1% level.