Math 341. Test No. 1. September 25, 1998.

Name: ....

1. (12 points) Two events A and B are such that $P(A) = .5$, $P(B) = .55$, and $P(A \cup B) = .75$.

(a) Find $P(A|B)$

(b) Find $P(A^c|B)$ [note: A^c means the event complementary to A]

(c) Find $P(A|A \cup B)$

(d) Find $P(A \cap B|A \cup B)$

Answer:

By the inclussion–exclussion formula

$$.75 = P(A \cup B) = P(A) + P(B) - P(A \cap B) = .5 + .55 - P(A \cap B)$$

we get $P(A \cap B) = 0.3$. We also have

$$0.3 + P(A^c \cap B) = P(A \cap B) + P(A^c \cap B) = P(B) = .55.$$

So, $P(A^c \cap B) = .25$.

(a) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.3}{.55} = .5454$

(b) $P(A^c|B) = \frac{P(A^c \cap B)}{P(B)} = \frac{0.25}{.55} = .4545$

Alternatively, $P(A^c|B) = 1 - P(A|B) = .4546$

(c) $P(A|A \cup B) = \frac{P(A)}{P(A \cup B)} = \frac{.5}{.75} = .666$

(d) $P(A \cap B|A \cup B) = \frac{P(A \cap B)}{P(A \cup B)} = \frac{.3}{.75} = .4$

2. (6 points). A bin contains twelve thermostats, of which five open at 85^o; and seven open at 90^o. If four are selected at random, what is the probability that there are two of each type in the sample? Give a numerical answer in terms of a simple fraction or a decimal number.

Answer: To find the cardinality in the event, we have to choose 2 thermostats from the twelve thermostats which open at 85^o; and 2 from the seven open at 90^o. So, the cardinality of this event is $\binom{5}{2}\binom{7}{2}$. The cardinality of the sample space is $\binom{12}{4}$. So, the probability is

$$\frac{\binom{5}{2}\binom{7}{2}}{\binom{12}{4}} = \frac{210}{495} = .4242$$

3. (10 points) A certain medical test has the following reliability: If a person has the disease, the test will show positive with probability .8. If the person tested does not have the

disease, the test will show negative with probability .9. Overall, 10 % of the population has the disease. If a person tests negative, what is the probability that the person is actually free of the disease?

Answer: Let $B_1 = \{\text{person has the disease}\}$, let $B_2 = \{\text{person is free of disease}\}$ and let $A = \{\text{test is negative}\}$. By the Bayes formula

$$\Pr(B_2|A) = \frac{\Pr(B_2)\Pr(A|B_2)}{\Pr(B_1)\Pr(A|B_1) + \Pr(B_2)\Pr(A|B_2)} = \frac{.9 \cdot .9}{(.1 \cdot .2) + (.9 \cdot .9)} = .9759.$$

4. (12 points) A certain data transmission system transmits units of data called bits. Bits are either correctly transmitted or incorrectly transmitted. The probability of correctly transmitting a bit is .999. This probability is the same for all bits and the errors are independent. A total of 5,000 bits is transmitted.

a) What is the probability that there are more than six errors?
b) What is the probability that there are exactly five errors?
c) What is the expected number of errors?
d) What is the variance of the number of errors?

Answer: We have a binomial distribution with n=5000 and p=.001. Since n is so large and p is so small, we can use the Poisson approximation. We can assume that X has a Poisson distribution with $\lambda = np = 5$.

a)

$$\Pr(X > 6)$$

$$= 1 - \Pr(X=0) - \Pr(X=1) - \Pr(X=2) - \Pr(X=3) - \Pr(X=4) - \Pr(X=5) - \Pr(X=6)$$

$$= 1 - (e^{-5} - e^{-5} \cdot 5 - \frac{e^{-5} \cdot 5^2}{2!} - \frac{e^{-5} \cdot 5^3}{3!} - \frac{e^{-5} \cdot 5^4}{4!} - \frac{e^{-5} \cdot 5^5}{5!} - \frac{e^{-5} \cdot 5^6}{6!} = .2377$$

Or using the binomial distribution

$$\Pr(X > 6) = 1 - \binom{5000}{0}.999^{5000} - \binom{5000}{1}.001^1 \cdot .999^{4999} - \binom{5000}{2}.001^2 \cdot .999^{4998}$$

$$-\binom{5000}{3}.001^3 \cdot .999^{4997} - \binom{5000}{4}.001^4 \cdot .999^{4996} - \binom{5000}{5}.001^5 \cdot .999^{4995} - \binom{5000}{6}.001^6 \cdot .999^{4994}$$

$$= .2377$$

b) $\Pr(X = 5) = e^{-5} \cdot 5^5/120 = .17547$, or using the binomial distribution

$$\Pr(X = 5) = \binom{5000}{5}.001^5 \cdot .999^{4995} = .1755$$

c) $E[X] = np = (5000)(.001) = 50$
d) $\mathrm{Var(X)} = \mathrm{np(1-p)} = (5000)(.001)(.999) = 4.995$

5. (15 points) A radar complex consists of two units operating independently. The probability that one of the units detects an oncoming missile is 0.95, and the probability that the other unit detects the missile is 0.90. Find the probability that:
a) both units will detect the missile.
b) at least one unit will detect the missile.
c) neither of the units will detect the missile.

Answer: Let $B_1 = \{$ the first unit detects the missile$\}$ let
$B_2 = \{$the second unit detects the missile$\}$
(a) Since B_1 and B_2 are independent,

$$P(B_1 \cap B_2) = P(B_1)P(B_2) = 0.95 \cdot 0.9 = .855$$

(b) Using

$$P(B_1 \cup B_2) = P(B_1) + P(B_2) - P(B_1 \cap B_2) = 0.95 + 0.9 - .855 = .995$$

(c) By the De Morgan formula,

$$P(B_1^c \cap B_2^c) = 1 - P(B_1 \cup B_2) = .005$$

6. (10 points) A certain random variable has a cumulative distribution function (cdf) $F_X(x)$ given by the following formula:

$$F(x) = \begin{cases} 0, & \text{if } -\infty < x < 0; \\ \frac{1}{8}, & \text{if } 0 \le x < 2; \\ \frac{(x-2)^2}{4} + \frac{1}{8}, & \text{if } 2 \le x < 3; \\ \frac{1}{2} + \frac{x-3}{2}, & \text{if } 3 \le x < 4; \\ 1, & \text{if } 4 \le x < \infty; \end{cases}$$

For this random variable X find the following probabilities:
a) $P(X \le 0)$
b) $P(0 < X \le 1)$
c) $P(1.5 < X \le 2.5)$
d) $P(1.5 \le X \le 2.5)$
e) $P(1 \le X < 3)$
f) $P(2 < X \le 5)$

g) $P(X = 3)$
h) $P(X < 3)$
i) $P(X < 3.5)$
j) $P(X > 3.5)$

Answer:
a) $P(X \leq 0) = F_X(0) = \frac{1}{8}$
b) $P(0 < X \leq 1) = F_X(1) - F_X(0) = \frac{1}{8} - \frac{1}{8} = 0$
c) $P(1.5 < X \leq 2.5) = F_X(2.5) - F_X(1.5) = (\frac{(2.5-2)^2}{4} + \frac{1}{8}) - \frac{1}{8} = \frac{1}{16}$
d) $P(1.5 \leq X \leq 2.5) = F_X(2.5) - F_X(1.5-) = (\frac{(2.5-2)^2}{4} + \frac{1}{8}) - \frac{1}{8} = \frac{1}{16}$
e) $P(1 \leq X < 3) = F_X(3-) - F_X(1-) = (\frac{(3-2)^2}{4} + \frac{1}{8}) - \frac{1}{8} = \frac{1}{4}$
f) $P(2 < X \leq 5) = F_X(5) - F_X(2) = 1 - (\frac{(2-2)^2}{4} + \frac{1}{8}) = \frac{7}{8}$
g) $P(X = 3) = F_X(3) - F_X(3-) = (\frac{1}{2} + \frac{3-3}{2}) - (\frac{(3-2)^2}{4} + \frac{1}{8}) = \frac{1}{8}$
h) $P(X < 3) = F_X(3-) = \frac{(3-2)^2}{4} + \frac{1}{8} = \frac{3}{8}$
i) $P(X < 3.5) = F_X(3.5-) = \frac{1}{2} + \frac{3.5-3}{2} = \frac{3}{4}$
j) $P(X > 3.5) = 1 - F_X(3.5) = 1 - (\frac{1}{2} + \frac{3.5-3}{2}) = \frac{1}{4}$

7. (10 points) A lot of 500 items contains 10 defective items. A random sample of size 20 is taken *without replacement.*

a) What is the probability that the sample contains at most 1 defective item?

b) What is the probability of drawing exactly three defective items if the sample is drawn *with replacement*?

Answer:
Let X be the numer of defective items in the sample.
(a) Without replacement, the distribution of X is hypergeometric

$$\Pr(X \leq 1) = \Pr(X = 0) + \Pr(X = 1) = \frac{\binom{10}{0}\binom{490}{20}}{\binom{500}{20}} + \frac{\binom{10}{1}\binom{490}{19}}{\binom{500}{20}}$$

(b) With replacement, X has a binomial distribution with parameters $n = 20$ and $p = 10/500 = .02$.

$$\Pr(X = 3) = \binom{20}{3} .02^3 (1 - 0.02)^{17}$$

8. (10 points) A certain random variable has a moment generating function given by $M(t) = \frac{e^t + e^{-t}}{2}$. Find the mean and variance for this random variable.

Answer: $M'(t) = \frac{e^t - e^{-t}}{2}$, $M'(0) = 0$. So the mean is 0. We also have that $M''(t) = \frac{e^t + e^{-t}}{2}$, $M''(0) = 1$. So $E[X^2] = 1$ and $\mathrm{Var}(X) = 1$.

9. (15 points) A certain computer system transmits data to another computer. The data are transmitted in groups called packets. Each packet is transmitted by transmitting a total of 50,000 data bits. The probability of an error in transmitting any given bit is 10^{-5}. Furthermore the events that an error occurs on different bits are independent. An error correction scheme is used which can correct at most two errors, but if more than two errors occur, the packet must be retransmitted. Suppose that 1000 packets of such data are transmitted. What is the expected value of the number of packets, which must be retransmitted? Give a numerical answer.

Answer: Let X be the numbers of errors in a packett. X has a binomial distribution with $n = 50,000$ and $p = 10^{-5}$. Since n is so large and p is so small, we can use the Poisson approximation. We can assume that X has a Poisson distribution with $\lambda = np = .5$

$$\Pr(X \leq 2) = \Pr(X = 0) + \Pr(X = 1) + P(X = 2) = e^{-.5} + e^{-.5}.5 + \frac{e^{-.5}.5^2}{2!} = .985612$$

Alternatively, we can use the binomial distribution:

$$\Pr(X \leq 2) = .99999^{50000} + \binom{50000}{1}.99999^{49999} \cdot .00001 + \binom{50000}{2}.99999^{49998} \cdot .00001^2 = .985613$$

So, the probability that a packett needs to be retransmitted is .014387. In 1000 packets, the expected value of the number of packets, which must be retransmitted is $1000 \cdot .014387. = 14.387$