MATH. 341. Test No. 3. November 29, 2000.

Name: ....

1. (10 points). The scores of a reference population in the Wechsler Intelligence Scale for Children (WISC) are normally distributed with $\mu = 100$ and $\sigma = 15$.

(a) What is the probability of WISC score above 130?

(b) What score should a children achieve on the WISC in order to fall in the top 5 % of the population?

Solution:

(a) Standardizing the normal random variable and using the normal table, we get that

$$P(X \geq 130) = P\left(N(0,1) \geq \frac{130-100}{15}\right) = P(N(0,1) \geq 2) = 1-\Phi(2) = 1-.977250 = .02275.$$

So, the ansswer to (a) is 0. 02275.

(b) Let a be the socre a children should achieve on the WISC in order to fall in the top 5 % of the population. Then,

$$.95 = P(X \leq a) = P\left(N(0,1) \leq \frac{a-100}{15}\right).$$

So, $\Phi^{-1}(.095) = \frac{a-100}{15}$. By interpolation

$$\Phi^{-1}(.095) = \left(1 - \frac{.95 - .945201}{0.955435 - 0.94201}\right) \cdot 1.6 + \left(\frac{.95 - .945201}{0.955435 - 0.94201}\right) \cdot 1.7$$

$$= (.5311) \cdot (1.6) + (.4689) \cdot (1.7) = 1.64689.$$

So, $1.64689 = \frac{a-100}{15}$ which implies $a = 100 + (15) \cdot (1.64689) = 124.70335$. Hence, the answer to (b) is 124.70335.

2. (15 points) One thousand independent rolls of a fair die will be made. Let X be the number of sixes, which will appear. Find the probability that $160 \leq X \leq 190$.

Solution: X has a binomial distribution with $n = 1000$ and $p = \frac{1}{6}$. So, $E[X] = np = 1000 \cdot \frac{1}{6} = 166.6667$ and $\mathrm{V}(X) = np(1-p) = 1000 \cdot \frac{1}{6} \cdot (1 - \frac{1}{6}) = 138.8889$. By the central limit theorem, using continuity correction:

$$P(160 \leq X \leq 190) = P(159.5 \leq X \leq 190.5)$$

$$\simeq P\left(\frac{159.5 - 166.6667}{\sqrt{138.8889}} \leq N(0,1) \leq \frac{190.5 - 166.6667}{\sqrt{138.8889}}\right) = P(-.6081 \leq N(0,1) \leq 2.2023).$$

By interpolation

$$\Phi(2.0223) = (1 - .223) \cdot \Phi(2) + (.223) \cdot \Phi(2.1) = .777 \cdot .977250 + (.223) \cdot .982136 = .9783,$$

$$\Phi(0.6081) = (1 - .081) \cdot \Phi(0.6) + (.081) \cdot \Phi(0.7) = .919 \cdot .725747 + (.081) \cdot .758036 = .7283$$

and

$$\Phi(-0.6081) = 1 - .7283 = .2716$$

So,

$$P(160 \le X \le 190) = 0.9783 - 0.2716 = 0.7067$$

3. (15 points). Let X, Y be random variables with a joint pdf

$$f(x,y) = \begin{cases} 6x & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find
(a) The marginal pdf of X, and $E\{X\}$.
(b) The marginal pdf of Y, and $E\{Y\}$.
(c) $E\{XY\}$
(d) The covariance of X and Y.

Solution:

Solution:
For $0 < x < 1$,

$$f_X(x) = \int_x^1 6x\,dy = 6x(1-x) = 6x - 6x^2$$

$f_X(x) = 0$, else.

$$E[X] = \int_0^1 x(6x - 6x^2)\,dx = \frac{6}{3} - \frac{6}{4} = \frac{1}{2}.$$

For $0 < y < 1$,

$$f_Y(y) = \int_0^y 6x\,dx = 3x^2 \mid_0^y = 3y^2$$

$f_Y(y) = 0$, else.

$$E[Y] = \int_0^1 y3y^2\,dx = \frac{3}{4}.$$

To find the covariance, we find first

$$E[XY] = \int_0^1 \int_x^1 xy6x\,dy\,dx = \int_0^1 3x^2y^2|_x^1\,dx = \int_0^1 (3x^2 - 3x^4)\,dx = 1 - \frac{3}{5} = \frac{2}{5}.$$

Hence,

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{2}{5} - \frac{1}{2}\cdot\frac{3}{4} = \frac{1}{40}.$$

4. (10 points). Given a random sample of size n from a distribution having the density

$$f(x,y) = \begin{cases} \frac{3x^2 e^{-\frac{x^3}{\theta}}}{\theta} & \text{if } 0 < x \\ 0 & \text{otherwise} \end{cases}$$

Find the maximum likelihood estimator of θ.

Solution: The likelihood is

$$L(\theta) = \prod_{j=1}^{n} \frac{3x_j^2 e^{-x_j^2/\theta}}{\theta} = \frac{3^n \cdot \prod_{j=1}^{n} x_j^2 \cdot e^{-\frac{\sum_{j=1}^{n} x_j^2}{\theta}}}{\theta^n}.$$

Hence,

$$\ln(L(\theta)) = n\ln(3) + 2\sum_{j=1}^{n}\ln(x_j) - \frac{\sum_{j=1}^{n} x_j^2}{\theta} - n\ln(\theta).$$

Solving,

$$\frac{\partial \ln(L(\theta))}{\partial\theta} = \frac{\sum_{j=1}^{n} x_j^2}{\theta^2} - \frac{n}{\theta} = 0$$

we get

$$\hat{\theta} = \frac{\sum_{j=1}^{n} x_j^2}{n}.$$

5. (20 points). A tobacco company claims that the amount of nicotine in its cigarettes is a random variable having a normal distribution with mean 2.2 mg. and standard deviation .7 mg. The mean of a random sample of $n = 100$ cigarettes was 2.3 mg.

(a) Test at the level $\alpha = .05$, the hypothesis $H_0 : \mu = 2.2$ against the one sided alternative $H_0 : \mu > 2.2$.

(b) Calculate the p-value of the test. (Use the attached tables of the normal distribution to give a numerical answer.)

Solution: We reject H_0 if $\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} \geq z_{0.95}$. We have that

$$\frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} = \frac{\sqrt{100}(2.3-2.2)}{0.7} = 1.4285$$

By interpolation,

$$\Phi^{-1}(.95) = \left(1 - \frac{.95 - .945201}{0.955435 - 0.94201}\right)\cdot 1.6 + \left(\frac{.95 - .945201}{0.955435 - 0.94201}\right)\cdot 1.7$$

$$= (.5311) \cdot (1.6) + (.4689) \cdot (1.7) = 1.64689.$$

The decision in the test is to accept H_0.

The p-value of the test is

$$P(N(0,1) \geq 1.4285) = 1 - \Phi(1.4285)$$

By interpolation,

$$\Phi(1.4285) = (1 - .285) \cdot \Phi(1.4) + .285 \cdot \Phi(1.5)$$

$$= (.715) \cdot (.919243) + (.285) \cdot (.933193) = .9232.$$

So, the p-value is $1 - .9232 = .0767$.

6. (20 points). A gun–like apparatus has recently been designed to replace needles in administering vaccines. The apparatus can be set to inject different amounts of the serum, but because of random fluctuations the actual amount injected is normally distributed with a mean equal to the setting and with an unknown variance σ^2. It has been decided that the apparatus would be too dangerous to use if σ exceeds 10. If a random sample of 50 injections resulted in a sample standard deviation of .12, should use the new apparatus be discontinued? What are the null and alternative hypothesis? Would you reject the null hypothesis if the significance level is (a) $\alpha = 0.10$? (b) $\alpha = 0.05$? (c) $\alpha = 0.01$?

Solution: The null hypothesis is $H_0 : \sigma = 0.10$ and the alternative hypothesis is $H_a : \sigma > 0.10$. We reject H_0 is $\frac{(n-1)s^2}{\sigma_0^2} \geq \chi^2_{1-\alpha}(n-1)$. We have that

$$\frac{(n-1)s^2}{\sigma_0^2} = \frac{49 \cdot (.12)^2}{.10^2} = 70.56.$$

When $\alpha = .10$, $\chi^2_{1-\alpha}(n-1) = \chi^2_{.9}(49) = 62.0375$. So, we reject H_0 at the level $\alpha = .10$.
When $\alpha = .05$, $\chi^2_{1-\alpha}(n-1) = \chi^2_{.95}(49) = 66.3386$. So, we reject H_0 at the level $\alpha = .05$.
When $\alpha = .01$, $\chi^2_{1-\alpha}(n-1) = \chi^2_{.99}(49) = 74.9195$. So, we accept H_0 at the level $\alpha = .01$.

7. (10 points). Suppose that a random sample of $n = 25$ recently sold houses in a certain city resulted in a sample mean price of $122,000, with a sample standard deviation of $12,000. Give a 95 percent confidence interval for the mean price of all recently sold houses in this city.

Solution: The 95 % confidence interal is

$$\bar{x} \pm t_{1-\frac{\alpha}{2}}(n-1)\frac{s}{\sqrt{n}} = 122,00 \pm 2.06390 \cdot \frac{12000}{\sqrt{25}} = 122000 \pm 4953 = (117046, 126953).$$