MATH. 341. Test No. 4 December 3, 1999.

Name: ....

1. (20 points total) Suppose that $Y_1, \ldots, Y_n$ represent a random sample from a Gamma distribution with known shape parameter ν, and unknown scale parameter β .

(a) (5 points) What is the expected value $E(Y)$?

(b) (15 points) Use the method of moments to find an estimator for β.

Answer: We know that the gamma density is $f(x, \nu, \beta) = \frac{x^{\nu-1}e^{-x/\beta}}{\Gamma(\nu)\beta^{\nu}}$, if $x \geq 0$. We also know that $E[X] = \nu\beta$. So, $E(Y) = \nu\beta$.

To find the method of the moments estimators of β, we solve for β, the equation $M_1 = \nu\beta$ to get $\hat{\beta} = \frac{M_1}{\nu} = \frac{\bar{x}}{\nu}$.

2. (15 points) A linear model of the form $y = \beta_0 + \beta_1 x + \epsilon$ is assumed to describe a certain situation. In a sample of size n=50 we got $\sum_{i=1}^{n} Y_i = 150$, $\sum_{i=1}^{n} X_i = 0$, $\sum_{i=1}^{n} X_i Y_i = 350$ and $\sum_{i=1}^{n} X_i^2 = 250$. Find the least squares estimate of the slope and intercept.

Answer: We have that

$$\hat{\beta}_1 = \frac{S_{x,y}}{S_x^2} = \frac{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$= \frac{\frac{1}{n}\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \cdot \frac{1}{n}\sum_{i=1}^{n} y_i}{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\frac{1}{n}\sum_{i=1}^{n} x_i)^2} = \frac{\frac{350}{50} - \frac{0}{50}\cdot\frac{150}{50}}{\frac{250}{50} - (\frac{0}{50})^2} = \frac{7}{5} = 1.4$$

We also have that

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} = \frac{150}{50} - (1.4 \cdot 0) = 3.$$

3. (15 points) A random sample of size 100 is taken from a normal distribution, which has an unknown mean μ and known variance $\sigma^2 = 25$. The hypothesis $H_0 : \mu = 20$ is to be tested against the one sided alternative $H_a : \mu > 20$. The observed value of the sample mean $\bar{x}$ is 20.75. Calculate the p-value of the test. Use the attached tables of the normal distribution to give a numerical answer.

Answer: Since $E[\bar{x}] = \mu = 20$ and $\mathrm{V}(\bar{x}) = \frac{\sigma^2}{n} = \frac{25}{100} = .25$,

$$\mathrm{p-value} \ = \Pr\{\bar{x} \geq 20.75\} = \Pr\{\mathrm{N}(0,1) \geq \frac{20.75 - 20}{\sqrt{.25}}\}$$

$$= \Pr\{\mathrm{N}(0,1) \geq 1.5\} = 1 - .933193 = .066807$$

4. (20 points total) A random sample of $n = 50$ steel rods is taken. We wish to perform a one sample t-test of the hypothesis that the mean $\mu = 20$ vs the alternative hypothesis that

the μ is smaller than 20. The sample mean is 18.89. Also, the sample standard deviation $s = 1.12$. Answer the following questions:

(a) (9 points) Calculate the t value for the test.

(b) (3 points) The number of degrees of freedom for this test is?

(c) (4 points) Is the null hypothesis rejected at the 5% level?

(d) (4 points) Is the null hypothesis rejected at the 1% level?

Answer: The t–value for this test is $t(\text{sample}) = \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{18.89-20}{1.12/\sqrt{20}} = -4.4322$.

The degrees of freedom of this t distribution are $n - 1 = 49$. So, we use the normal distribution.

According to the tables, we reject the null hypothesis is $t(\text{sample}) \leq -t(\alpha)$.

When $\alpha = 5\%$, we reject the null hypothesis if $t(\text{sample}) \leq -t(.95) = -1.64673$. Note that by interpolation

$$\Phi^{-1}(.95) = \left(1 - \frac{.95 - .945201}{.955435 - .945201}\right)\Phi^{-1}(.955435) + \frac{.95 - .945201}{.955435 - .945201}\Phi^{-1}(.945201)$$

$$= (.5311) \cdot (1.6) + (.4689) \cdot (1.7) = 1.64673.$$

We reject the null hypothesis.

When $\alpha = 1\%$, we reject the null hypothesis if $t(\text{sample}) \leq -t(.99) = -2.32866$. Note that by interpolation

$$\Phi^{-1}(.99) = \left(1 - \frac{.99 - .989276}{.991802 - .989276}\right)\Phi^{-1}(.989276) + \frac{.99 - .989276}{.991802 - .989276}\Phi^{-1}(.991802)$$

$$= (.7134) \cdot (2.3) + (.2866) \cdot (2.4) = 2.32866.$$

We reject the null hypothesis.

5. (15 points) A random sample of size $n = 25$ from a normal distribution with unknown variance σ^2 gave the following results. The sample mean was 30 and the sample variance was 1.25. Give numerical answers for the following:

(a) (8 points) The confidence interval for the mean μ at the level of confidence $1-\alpha = .95$.

(b) (7 points) The confidence interval for the variance σ^2, at the level of confidence $1-\alpha = .90$.

Answer: We have that $1-\frac{\alpha}{2} = .975$ and the degrees of freedom are 24. So, $t_{1-\frac{\alpha}{2}} = 2.06390$ and the confidence interval of the mean is

$$\bar{x} \pm t_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} = 30 \pm (2.06390)\frac{\sqrt{1.25}}{\sqrt{25}} = 30 \pm .4615 = (29.5385, 30.4615).$$

The confidence interval for σ^2 is

$$\left(\frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}, \frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}(n-1)}\right) = \left(\frac{24 \cdot 1.25}{39.3641}, \frac{24 \cdot 1.25}{12.4012}\right) = (.7621, 2.4191).$$

6. (15 points) In $n = 200$ Bernoulli trials one observes $X = 156$ successes. Construct a large sample confidence interval, with level of confidence $1 - \alpha = .95$, for the probability of success p.

Answer: We have that $z_{1-\frac{\alpha}{2}} = 1.96$ and $\hat{p} = \frac{156}{200} = .78$. The large sample confidence interval for p is

$$\hat{p} \pm z_{1-\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = .78 \pm 1.96 \sqrt{\frac{.78 \cdot (1-.78)}{200}} = .78 \pm 0.0574 = (.7226, .8374)$$