Power of tests and the Neyman-Pearson lemma

Def The power of a test is the function

$$\text{power}(\theta) = P(\text{reject } H_0, \text{ when the parameter value is } \theta)$$

If $\theta \in H_0$, power(θ) = error

If $\theta \in H_a$: power$(\theta) = 1 -$ error. $= 1 - \beta(\theta)$

We want to choose the rejection region so that

(1) $\sup_{\theta \in \Theta_0} \text{power}(\theta) \leq \alpha$

(2) power(θ) as large as possible for $\theta \in H_a$.

Def A hypothesis is said to be a simple hypothesis if that hypothesis uniquely specifies the distribution from which the sample is taken. Ex: $H_0: \theta = \theta_0$

Any hypothesis that is not a simple hypothesis is called a composite hypothesis

Neyman-Pearson lemma

Consider the test of hypothesis $H_0: \theta = \theta_0$ versus $H_a: \theta = \theta_a$ based on a random sample $Y_1 - Y_n$ from a distribution with parameter θ. Let $L(\theta)$ denote the likelihood of the sample when the value of the parameter is θ.

Between all rejection regions such that

P(rejection region $| H_0) \leq \alpha$

the one with biggest P(rejection region $| H_a)$ is

$$RR = \left\{ \frac{L(\theta_0)}{L(\theta_a)} < K \right\}$$

Such a test is a most powerful test for H_0 versus H_a.

Given $H_0: \theta \in \Theta_0$ versus $H_a: \theta \notin \Theta_0$.

a level α test such that $\text{power}_\theta(\text{test})$ is maximized for each $\theta \notin \Theta_0$ is called a uniformly most powerful test.

In some situations there are uniformly most powerful test, for $H_0: \theta = \theta_0$ versus $H_a: \theta > \theta_0$

Consider the hypothesis testing problem

$H_0: \theta = \theta_0$ versus $H_a: \theta > \theta_0$

Suppose that for each $\theta_a > a$,

The rejection region $\frac{L(\theta_0)}{L(\theta_a)} \leq K$

is equivalent to $T(X_1 - X_n) \geq c$, for some fixed statistic $T(X_1 - X_n)$

Then, choose c such that

$\alpha = P_{\theta_0}(T(X_1 - - X_n) \geq c)$

The test who reject H_0 if $T(X_1 - X_n) \geq c$ is the α-level most powerfule test for testing $H_0: \theta = \theta_0$ versus $H_a: \theta = \theta_a$

Hence, the test who reject H_0 if $T(X_1 - X_n) \geq c$ is a uniformly most powerful test

Example 10.23 Suppose that $Y_1, \ldots, Y_n$ constitute a random sample from a normal distribution with unknown mean μ and known variance σ^2

We wish to test $H_0: \mu = \mu_0$ against $H_a: \mu > \mu_0$ for a specified constant μ_0. Find the uniform most powerful test significance level α.

$$f(y|\mu) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$$

$$L(\mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y_i-\mu)^2}{2\sigma^2}} = \frac{1}{(\sqrt{2\pi})^n \sigma^n} e^{-\frac{\sum (y_i-\mu)^2}{2\sigma^2}}$$

The most powerful test of $H_0: \mu = \mu_0$ versus $H_a^*: \mu = \mu_a$ where $\mu_a > \mu_0$ is given by

$$k > \frac{L(\mu_0)}{L(\mu_a)} = \frac{\dfrac{e^{-\frac{\sum (y_i-\mu_0)^2}{2\sigma^2}}}{(\sqrt{2\pi})^n \sigma^n}}{\dfrac{e^{-\frac{\sum (y_i-\mu_a)^2}{2\sigma^2}}}{(\sqrt{2\pi})^n \sigma^n}} = e^{+\frac{\sum (y_i-\mu_a)^2}{2\sigma^2} - \frac{\sum (y_i-\mu_0)^2}{2\sigma^2}}$$

$$= e^{\frac{\sum (y_i-\bar{y})^2}{2\sigma^2} + \frac{n(\bar{y}-\mu_a)^2}{2\sigma^2} - \left(\frac{\sum (y_i-\bar{y})^2}{2\sigma^2} + \frac{n(\bar{y}-\mu_0)^2}{2\sigma^2}\right)}$$

$$= e^{\frac{n}{2\sigma^2}\left(\bar{y}^2 - 2\mu_a \bar{y} + \mu_a^2 - \bar{y}^2 + 2\mu_0 y - \mu_0^2\right)} = e^{\frac{n}{2\sigma^2}\left(2(\mu_0-\mu_a)\bar{y} + \mu_a^2 - \mu_0^2\right)}$$

$$\ln k \geqslant \frac{n}{2\sigma^2}\left(2(\mu_0-\mu_a)\bar{y} + \mu_a^2 - \mu_0^2 \right)$$

$$\frac{2\sigma^2}{n}\ln k - (\mu_a^2-\mu_0^2) \geqslant 2\,(\mu_0-\mu_a)\,\bar{y}$$

$$\frac{\frac{2\sigma^2 \ln k}{n} - (\mu_a^2-\mu_0^2)}{2(\mu_0-\mu_a)} \leq \bar{y}$$

or $\bar{y} \geqslant k'$, k' is chosen so that

$\alpha = P_{H_0}(\bar{y} \geq k')$ $\qquad \bar{y} \geqslant \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} = k'$

k' does not depend on $\mu_a > \mu_0$

So $\{\bar{y} \geqslant \mu_0 + \frac{z_\alpha \sigma}{\sqrt{n}}\}$ is the uniform most powerful region

to test $H_0: \mu=\mu_0$ versus $H_a: \mu>\mu_0$