<u>Example</u> (almost 10.22)

Let $Y_1 \ldots Y_n$ be a random sample from the density

$$f(y|\theta) = \begin{cases} \theta y^{\theta-1} & \text{if } 0<y<1 \\ 0 & \text{else} \end{cases}$$

Find the form of the most powerful test for $H_0: \theta = \theta_0$ versus $H_a: \theta = \theta_a$ where $\theta_a > \theta_0$

$$L(\theta) = \prod_{i=1}^{n} \theta Y_i^{\theta-1} = \theta^n \left(\prod Y_i\right)^{\theta-1}$$

Reject H_0 if

$$K > \frac{L(\theta_0)}{L(\theta_a)} = \frac{\theta_0^n \left(\prod Y_i\right)^{\theta_0-1}}{\theta_a^n \left(\prod Y_i\right)^{\theta_a-1}} = \frac{\theta_0^n}{\theta_a^n} \left(\prod Y_i\right)^{\theta_0-\theta_a}$$

$$\frac{K\theta_a^n}{\theta_0^n} \geq \left(\prod Y_i\right)^{\theta_0-\theta_a}$$

$$\ln\left(\frac{K\theta_a^n}{\theta_0^n}\right) \geq (\theta_0-\theta_a) \sum_{i=1}^{n} \ln Y_i$$

$$\frac{\ln\left(\frac{K\theta_a^n}{\theta_0^n}\right)}{\theta_0-\theta_a} \leq \sum_{i=1}^{n} \ln Y_i$$

The form of the most powerful is $\sum_{i=1}^{n} \ln Y_i \geq K'$

10.83

Let $Y_1 \ldots Y_n$ denote a random sample from the density

$$f(y|\theta) = \begin{cases} \frac{1}{2\theta^3} y^2 e^{-y/\theta}, & y > 0 \\ 0 & \text{else} \end{cases}$$

a. Find the rejection region for the most powerful test of $H_0 : \theta = \theta_0$ against $H_a : \theta = \theta_a$, assuming that $\theta_a > \theta_0$

b. Is the test given in (a) uniformly most powerful against the alternative $\theta > \theta_0$?

$$f(y|\theta) = \frac{y^2}{2\theta^3} e^{-y/\theta} \qquad L(\theta) = \frac{\prod Y_i^2 e^{-\sum Y_i/\theta}}{2^n \theta^{3n}}$$

Reject H_0 if

$$k > \frac{L(\theta_0)}{L(\theta_a)} = \frac{\dfrac{\prod Y_i^2 e^{-\sum Y_i/2\theta_0}}{2^n \theta_0^{3n}}}{\dfrac{\prod Y_i^2 e^{-\sum Y_i/2\theta_a}}{2^n \theta_a^{3n}}} = \frac{\theta_a^{3n}}{\theta_0^{3n}} e^{\frac{\sum Y_i}{2\theta_a} - \frac{\sum Y_i}{2\theta_0}}$$

$$\frac{k\,\theta_0^{3n}}{\theta_a^{3n}} \geq e^{\frac{\sum Y_i(\theta_0 - \theta_a)}{2\theta_0 \theta_a}} \qquad \ln\left(\frac{k\,\theta_0^{3n}}{\theta_a^{3n}}\right) \geq \frac{\sum Y_i (\theta_0 - \theta_a)}{2\theta_0 \theta_a}$$

$$k' = \frac{2\theta_a \theta_0}{(\theta_0 - \theta_a)} \ln\left(\frac{k\,\theta_0^{3n}}{\theta_a^{3n}}\right) \leq \sum_{i=1}^{n} Y_i$$

Reject H_0 if $\sum_{i=1}^{n} Y_i \geq k'$

10.90 Let $Y_1 \dots Y_n$ denote a random sample from a Bernoulli distribution with parameter p. Suppose that we are interested in testing $H_0: p = p_0$ versus $H_a: p = p_a$ where $p_0 < p_a$.

(a) Give the rejection region for the most powerful test of H_0 versus H_a

(b) Is the test derived in (a) uniformly most powerful for testing $H_0: p = p_0$ versus $H_a: p > p_0$?

$$f(y_i | p) = p^{y_i}(1-p)^{1-y_i}$$

$$L(p) = \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i} = p^{\sum y_i}(1-p)^{n-\sum y_i} = (1-p)^n\left(\frac{p}{1-p}\right)^{\sum y_i}$$

Reject H_0 if

$$k > \frac{L(p_0)}{L(p_a)} = \frac{(1-p_0)^n\left(\frac{p_0}{1-p_0}\right)^{\sum y_i}}{(1-p_a)^n\left(\frac{p_a}{1-p_a}\right)^{\sum y_i}}$$

or $$\frac{k(1-p_a)^n}{(1-p_0)^n} > \left(\frac{\frac{p_0}{1-p_0}}{\frac{p_a}{1-p_a}}\right)^{\sum y_i}$$

or $$\ln\left(\frac{k(1-p_a)^n}{(1-p_0)^n}\right) \geq \left(\sum y_i\right)\ln\left(\frac{\frac{p_0}{1-p_0}}{\frac{p_a}{1-p_a}}\right)$$

Now, if $p_0 < p_a$ then $\frac{p_0}{1-p_0} < \frac{p_a}{1-p_a}$

$f(x) = \frac{x}{1-x}$ is increasing in $(0,1)$

$$f'(x) = \frac{1\cdot(1-x) - x(-1)}{(1-x)^2} = \frac{1-x+1}{(1-x)^2} = \frac{1}{(1-x)^2}$$

So, $\ln\left(\dfrac{\frac{p_0}{1-p_0}}{\frac{p_a}{1-p_a}}\right) < 0$ and

$$\frac{\ln\left(\dfrac{k(1-p_a)^n}{(1-p_0)^n}\right)}{\ln\left(\dfrac{\frac{p_0}{1-p_0}}{\frac{p_a}{1-p_a}}\right)} \leq \sum_{i=1}^{n} Y_i$$

So, Reject H_0 if $\sum_{i=1}^{n} Y_i \geq k'$

(b) Yes, the test in (a) is uniformly most powerful for testing

$H_0: p = p_0$ versus $H_a: p > p_0$

10.84. Suppose that $Y_1, \dots Y_n$ denote a random sample from a population having an exponential distribution with mean θ.

(a) Derive the most powerful test for $H_0: \theta = \theta_0$ against $H_a: \theta = \theta_a$ where $\theta_a < \theta_0$

(b) Is the test derivated in (a) uniformly most powerful for testing $H_0: \theta = \theta_0$ against $H_a: \theta < \theta_0$

(a) $f(y|\theta) = \frac{1}{\theta} e^{-y/\theta}$, if $y > 0$

$$L(\theta) = \prod_{j=1}^{n} \frac{1}{\theta} e^{-y_j/\theta} = \frac{e^{-\sum y_j/\theta}}{\theta^n}, \text{ if } y_1 \dots y_n > 0$$

The Rejection region who maximizes the power a θ_a is

$$K > \frac{L(\theta_0)}{L(\theta_a)} = \frac{\frac{1}{\theta_0^n} e^{-\sum y_j/\theta}}{\frac{1}{\theta_a^n} e^{-\sum y_j/\theta_0}} = \frac{\theta_a^n}{\theta_0^n} e^{\sum y_j \left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right)}$$

$$\ln\left(\frac{K\theta_a^n}{\theta_0^n}\right) \geq \sum y_j \left(\frac{1}{\theta_0} - \frac{1}{\theta_a}\right) \qquad \frac{1}{\frac{1}{\theta_a} - \frac{1}{\theta_a}} \ln\left(\frac{K\theta_a^n}{\theta_0^n}\right) \geq \sum y_j$$

or $\sum_{j=1}^{n} y_j \leq c$ where $P_{\theta_0}\left(\sum_{j=1}^{n} y_j \leq c\right) = \alpha$.

(b) Yes the test who rejects H_0 if $\sum_{j=1}^{n} y_j \leq c$ is the uniformly most powerful test for $H_0: \theta = \theta_0$ against $H_a: \theta < \theta_0$