The likelihood ratio test

To test $H_0: \theta \in \Omega_0$ versus $H_a: \theta \in \Omega_a$, the likelihood ratio test rejects H_0 if

$$k \geq \lambda(x_1, \ldots x_n) = \frac{\sup_{\theta \in \Omega_0} L(\theta)}{\sup_{\theta \in \Omega} L(\theta)}$$

To find $\sup_{\theta \in \Omega} L(\theta)$, it suffices to find $L(\hat{\theta})$ where $\hat{\theta}$ is the mle

Let $Y_1 \ldots Y_n$ denote a random sample from $N(\mu, \sigma^2)$, where μ is unknown and σ^2 is known. Find the likelihood ratio test for $H_0: \mu = \mu_0$ versus $H_a: \mu \neq \mu_0$

$\Omega_0 = \{\mu_0\}$ $\quad \Omega = \{\mu : \mu \in \mathbb{R}\}$

$$L(\mu) = \prod_{i=1}^{n} \frac{e^{-\frac{(y_i-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} = \frac{e^{-\frac{\sum (y_i-\mu)^2}{2\sigma^2}}}{(\sqrt{2\pi})^n \sigma^n}$$

The likelihood ratio test rejects H_0 if

$$K \geq \lambda(x_1 \ldots x_n) = \frac{\sup_{\mu=\mu_0} L(\mu)}{\sup_{\mu} L(\mu)} = \frac{\dfrac{e^{-\frac{\sum (y_i-\mu_0)^2}{2\sigma^2}}}{(\sqrt{2\pi})^n \sigma^n}}{\sup_{\mu} \dfrac{e^{-\frac{\sum (y_i-\mu)^2}{2\sigma^2}}}{(\sqrt{2\pi})^n \sigma^n}} =$$

$$= \frac{e^{-\frac{\sum (y_i-\mu_0)^2}{2\sigma^2}}}{e^{-\frac{\sum (y_i-\bar{y})^2}{2\sigma}}} = e^{\frac{\sum (y_i-\bar{y})^2}{2\sigma^2} - \frac{\sum (y_i-\mu_0)^2}{2\sigma^2}}$$

Using that $\sum (y_i-\bar{y})^2 = \sum (y_i-\mu_0)^2 - n(\bar{y}-\mu_0)^2$

$$K \geq e^{-\frac{n(\bar{y}-\mu_0)^2}{2\sigma^2}} \qquad \ln K \geq -\frac{n}{2\sigma^2}(\bar{y}-\mu_0)^2$$

$$\frac{n}{2\sigma^2}(\bar{y}-\mu_0)^2 \geq -\ln K \qquad (\bar{y}-\mu_0)^2 \geq -\frac{2\sigma^2 \ln K}{n}$$

$$|\bar{y}-\mu_0| \geq \sqrt{-\frac{2\sigma^2 \ln K}{n}}$$

Let X have a Poisson pdf with parameter θ.
We want to use a random sample of size n to test $H_0: \theta = 1$ versus $H_a: \theta \neq 1$.

(a) Find the rejection region for the likelihood ratio test.

(b) Show that this test can be based on $\bar{x}$

$$f(x,\theta) = e^{-\theta} \frac{\theta^x}{x!} \qquad L(\theta) = \prod_{i=1}^{n} e^{-\theta} \frac{\theta^{x_i}}{x_i!} = e^{-\theta n} \frac{\theta^{\sum x_i}}{\prod x_i!}$$

Reject H_0 if

$$k \geq \lambda(x_1, \ldots, x_n) = \frac{\sup_{\theta=1} L(\theta)}{\sup_{\theta} L(\theta)} = \frac{\frac{e^{-n}}{\prod x_i!}}{\frac{e^{-\theta n} \theta^{\sum x_i}}{\prod x_i!}} = \frac{\frac{e^{-n}}{\prod x_i}}{\frac{e^{-n\bar{x}} \bar{x}^{n\bar{x}}}{\prod x_i!}}$$

$$= \frac{e^{-n} e^{n\bar{x}}}{(\bar{x})^{n\bar{x}}} \quad \text{or} \quad k^{1/n} \geq \frac{e^{\bar{x}}}{e\, \bar{x}^{\bar{x}}} = \frac{1}{e} \left(\frac{e}{\bar{x}}\right)^{\bar{x}}$$

Let $Y_1 \ldots Y_n$ denote a random sample from $N(\mu, \sigma^2)$ where μ is unknown and σ^2 is unknown. Find the likelihood ratio test for testing $H_0: \mu = \mu_0$ versus $H_a: \mu \neq \mu_0$

$\Omega_0 = \{ (\mu_0, \sigma^2) : \sigma^2 > 0 \}$

$\Omega = \{ (\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 > 0 \}$

$$L(\mu, \sigma^2) = \prod_{j=1}^{n} \frac{e^{\frac{-(y_j-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} = \frac{e^{-\frac{\sum (y_j-\mu)^2}{2\sigma^2}}}{(\sqrt{2\pi})^n \sigma^n}$$

The likelihood ratio test rejects H_0 if

$$k \geq \lambda(y_1 \ldots y_n) = \frac{\sup_{\sigma^2} L(\mu_0, \sigma^2)}{\sup_{\mu, \sigma^2} L(\mu, \sigma^2)} = \frac{\sup_{\sigma^2 > 0} \frac{e^{-\frac{\sum (y_j-\mu_0)^2}{2\sigma^2}}}{\sigma^n}}{\sup_{\mu, \sigma^2} \frac{e^{-\frac{\sum (y_j-\mu)^2}{2\sigma^2}}}{\sigma^n}}$$

Since $\sum (y_j-\mu)^2 = \sum (y_j - \bar{y})^2 + n(\bar{y}-\mu)^2)$

$$\sup_{\mu} \left(-\sum (y_j-\bar{y})^2\right) = \sup_{\mu} \left(-\sum (y_j-\bar{y})^2 - n(\bar{y}-\mu)^2\right) = -\sum (y_j-\bar{y})^2$$

$$k \geq \frac{\sup_{\sigma^2 \geq 0} \frac{e^{-\frac{\sum (y_j-\mu_0)^2}{2\sigma^2}}}{\sigma^n}}{\sup_{\sigma^2 > 0} \frac{e^{-\frac{\sum (y_j-\bar{y})^2}{2\sigma^2}}}{\sigma^n}}$$

Now, $\sup_{\sigma^2 > 0} \frac{e^{-a/\sigma^2}}{\sigma^n} = \left(\frac{n}{2a}\right)^{n/2} e^{-n/2}$