Consider $f(\sigma) = \dfrac{e^{-\frac{a}{\sigma^2}}}{\sigma^n}$

$$f'(\sigma) = e^{-\frac{a}{\sigma^n}}\left(\frac{2a}{\sigma^3}\frac{1}{\sigma^n} - \frac{n}{\sigma^{n+1}}\right) = 0 \qquad \sigma^2 = \frac{2a}{n}.$$

$$\underset{\sigma^2>0}{\sup} \frac{e^{-a/\sigma^2}}{\sigma^n} = f\left(\sqrt{\frac{2a}{n}}\right) = \frac{e^{-a/\left(\frac{2a}{n}\right)}}{\left(\frac{2a}{n}\right)^{n/2}} = \left(\frac{n}{2a}\right)^{n/2} e^{-n/2}$$

So

$$K \geq \frac{\left(\frac{n}{2\sum(y_i-\mu_0)^2}\right)^{n/2} e^{-n/2}}{\left(\frac{n}{2\sum(y_i-\bar{y})^2}\right)^{n/2} e^{-n/2}} = \left(\frac{\sum(y_i-\bar{y})^2}{\sum(y_j-\mu_0)^2}\right)^{n/2} = \left(\frac{\sum(y_i-\bar{y})^2}{\sum(y_i-\bar{y})^2 + n(\bar{y}-\mu_0)^2}\right)^{n/2}$$

$$K^{2/n} \geq \frac{1}{1 + \frac{n(\bar{y}-\mu_0)^2}{\sum(y_i-\bar{y})^2}} \qquad \frac{n(\bar{y}-\mu_0)^2}{\sum(y_i-\bar{y})^2} \geq \frac{1}{K^{2/n}} - 1$$

The t-test reject H_0 if

$$\frac{\sqrt{n}\,|\bar{y}-\mu_0|}{s} \geq t_{\frac{\alpha}{2}} \quad \text{or} \quad \frac{n(\bar{y}-\mu_0)^2}{\sum(y_j-\bar{y})^2} \geq \frac{t^2_{\alpha/2}}{n-1}$$

Example 10.24

Suppose that $Y_1 \ldots Y_n$ constitute a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. Find the likelihood ratio test for testing $H_0: \mu = \mu_0$ versus $H_a: \mu > \mu_0$

$$L(\mu, \sigma^2) = \prod_{j=1}^{n} \frac{e^{-\frac{(y_j-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} = \frac{e^{-\frac{\sum (y_j-\mu)^2}{2\sigma^2}}}{(\sqrt{2\pi})^n \sigma^n}$$

The likelihood ratio test rejects H_0 if

$$K \geq \lambda(y_1 \ldots y_n) = \frac{\sup_{\sigma^2 > 0} L(\mu_0, \sigma^2)}{\sup_{\substack{\mu \geq \mu_0 \\ \sigma^2 > 0}} L(\mu, \sigma^2)} = \frac{\sup_{\sigma^2 > 0} \frac{e^{-\frac{\sum (y_j-\mu_0)^2}{2\sigma^2}}}{\sigma^n}}{\sup_{\mu \geq \mu_0} \sup_{\sigma^2 \geq 0} \frac{e^{-\frac{\sum (y_j-\mu)^2}{2\sigma^2}}}{\sigma^n}}$$

$$= \frac{\left(\frac{n}{2 \sum (y_j-\mu_0)^2}\right)^{n/2} e^{-n/2}}{\sup_{\mu \geq \mu_0} \left(\frac{n}{2 \sum (y_j-\mu)^2}\right)^{n/2} e^{-n/2}} = \inf_{\mu \geq \mu_0} \left(\frac{\sum (y_j-\mu)^2}{\sum (y_j-\mu_0)^2}\right)^{n/2}$$

$$= \inf_{\mu \geq \mu_0} \left(\frac{\sum (y_j-\bar{y})^2 + n(\bar{y}-\mu)^2}{\sum (y_j-\mu_0)^2}\right)^{n/2}$$

If $\bar{y} \geq \mu_0$, $\lambda(y_1 \ldots y_n) = \left(\frac{\sum (y_j-\bar{y})^2}{\sum (y_j-\mu_0)^2}\right)^{n/2}$

If $\bar{y} \leq \mu_0$ $\lambda(y_1 \ldots y_n) = 1$

So, we take $\kappa < 1$, and reject for $\bar{y} > \mu_0$ with

$$\kappa^{2/n} \geqslant \frac{\sum (y_i - \bar{y})^2}{\sum (y_i - \bar{y})^2 + n(\bar{y} - \mu_0)^2} = \frac{1}{1 + \frac{n(\bar{y} - \mu_0)^2}{\sum (y_i - \bar{y})^2}}$$

$$\frac{\sqrt{n}(\bar{y} - \mu_0)}{\sqrt{\sum (y_i - \bar{y})^2}} \geqslant \frac{1}{\kappa^{2/n}} - 1$$

which is equivalent to

$$\bar{y} \geqslant \mu_0 + t_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

10.93. Let $Y_1, \dots, Y_n$ denote a random sample from a normal distribution with unknown mean μ and variance σ^2. Show that the LRT for testing $H_0: \sigma^2 = \sigma_0^2$ against $H_a: \sigma^2 > \sigma_0^2$

is $\frac{\sum (Y_i - \bar{Y})^2}{\sigma_0^2} \geq \chi^2_\alpha(n-1)$

$\Omega = \{(\mu, \sigma^2): \sigma^2 \geq \sigma_0^2\}$

$\Omega_0 = \{(\mu, \sigma_0^2): \mu \in \mathbb{R}\}$

$$L(\mu, \sigma^2) = \frac{e^{-\frac{\sum (y_i - \mu)^2}{2\sigma^2}}}{(\sqrt{2\pi})^n \sigma^n}$$

The LRT rejects H_0 if

$$k \geq \frac{\sup_{\mu} \frac{e^{-\frac{\sum (y_i-\mu)^2}{2\sigma_0^2}}}{\sigma_0^n}}{\sup_{\sigma^2 \geq \sigma_0} \sup_{\mu} \frac{e^{-\frac{\sum (y_i-\mu)^2}{2\sigma^2}}}{\sigma^n}} = \frac{\frac{e^{-\frac{\sum (y_i-\bar{y})^2}{2\sigma_0^2}}}{\sigma_0^n}}{\sup_{\sigma^2 \geq \sigma_0^2} \frac{e^{-\frac{\sum (y_i-\bar{y})^2}{2\sigma^2}}}{\sigma^n}} = \lambda(y_1 - y_n)$$

If $\sigma_0^2 \geq \frac{\sum (y_i - \bar{y})^2}{n}$, $\lambda(y_1 - y_n) = 1$

If $\frac{\sum (y_i - \bar{y})^2}{n} > \sigma_0^2$, $\lambda(y_1 - y_n) = \frac{\frac{e^{-\frac{\sum (y_i-\bar{y})^2}{2\sigma_0^2}}}{\sigma_0^n}}{\left(\frac{n}{\sum (y_i - \bar{y})^2}\right)^{n/2} e^{-n/2}}$

$$= \left(\frac{\sum (y_i-\bar{y})^2}{n\sigma_0^2}\right)^{n/2} e^{\frac{n}{2} - \frac{\sum (y_i-\bar{y})^2}{2\sigma_0^2}} = e^{n/2}\left(\frac{\sum (y_i-\bar{y})^2}{n\sigma_0^2} e^{-\frac{\sum (y_i-\bar{y})^2}{n\sigma_0^2}}\right)^{n/2}$$

Let $f(x) = e^{-x}x$, $x > 1$, $f'(x) = e^{-x}(1-x)$, f is decreasing in $(1, \infty)$

So, we reject if $\frac{\sum (y_i - \bar{y})^2}{n\sigma_0^2} \geq c$