Calculating the type II error probabilities and the sample size for the z-test

$\alpha =$ type I error $= P_{H_0}($reject $H_0)$

$\beta =$ type II error $= P_{H_a}($ do not reject $H_0)$

Suppose that to test $H_0: \mu = \mu_0$ versus $H_a: \mu = \mu_a$; $\mu_a > \mu_0$

Find the sample size so that type error I is α and the type error II is β. Assume σ is known

$$\alpha = P_{\mu_0}(\bar{y} > k) = P\left(N(0,1) \geq \frac{\sqrt{n}(k-\mu_0)}{\sigma}\right)$$

$$\beta = P_{\mu_a}(\bar{y} < k) = P\left(N(0,1) \leq \frac{\sqrt{n}(k-\mu_a)}{\sigma}\right) = P\left(N(0,1) \geq \frac{\sqrt{n}(\mu_a - k)}{\sigma}\right)$$

$$z_\alpha = \frac{\sqrt{n}(k-\mu_0)}{\sigma}$$

$$z_\beta = \frac{\sqrt{n}(\mu_a - k)}{\sigma}$$

$$z_\alpha + z_\beta = \frac{\sqrt{n}(\mu_a - \mu_0)}{\sigma}$$

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_a - \mu_0)^2}$$

A manufacturer of light bulbs claims that the life of the bulbs is normally distributed with mean 800 hr and standard deviation 40 h.

Before buying a large lot, a buyer tests 30 of the bulbs and finds an average life of 789 h.

a. Test the hypothesis $H_0: \mu = 800$ against the alternative $H_a: \mu < 800$ using a test of level 5 %

b. Find the probability of a type II error for the alternative $H_a: \mu = 790$

a. Reject H_0, if $\bar{Y} \leq \theta_0 - z_\alpha \frac{\sigma}{\sqrt{n}} = 800 - 1.645 \frac{(40)}{\sqrt{30}} = 791.05$

We reject H_0

b.

type II error $= P_{H_a}(\text{accept } H_0) = P(\bar{Y} \geq 791.05)$

$$= P\left(\frac{\bar{Y} - 790}{\sigma / \sqrt{n}} \geq \frac{791.05 - 790}{40 / \sqrt{30}} \right) = P(N(0,1) \geq 0.14)$$

$= 0.4443$

10.27 Refer to Exercise 10.9. If the voltage falls as low as 128, serious consequences may result. For testing $H_0: \mu = 130$ versus $H_a: \mu = 128$ find the probability β of a type II error for the rejection region used in Exercise 10.9 $\quad \alpha = 0.05$

To test $H_0: \mu = 130$ versus $H_a: \mu < 130$

We reject H_0 if $\bar{Y} \le \theta_0 - \frac{z_\alpha S}{\sqrt{n}} = 130 - \frac{(1.645)\,2.1}{\sqrt{40}} = 129.45$

type II error $=$ P(accept H_0, H_a is true) $= P(\bar{Y} \geq 129.45 \mid \mu = 128)$

$$= P\left(\frac{\bar{Y}-\mu}{s/\sqrt{n}} \geq \frac{129.45-128}{2.1/\sqrt{40}}\right) = P(N(0,1) \geq 4.37) = 0.00$$

10.20. A manufacturer claimed that at least 20% of the public preferred her product. A sample of 100 persons is taken to check her claim. With $\alpha = 0.05$, how small would the sample percentage have to be before the claim could legitimately be refuted?

$H_0: p = 0.20 \qquad H_a: p < 0.20$

Reject H_0 if $\quad \frac{Y}{n} \le p_0 - z_\alpha \sqrt{\frac{1}{n} p_0(1-p_0)}$

$= 0.20 - 1.645\sqrt{\frac{1}{100}(0.20)(0.80)} = 0.1342$

If $p = 0.10$, what is the probability of type II error?

type II error $= P(\text{accept } H_0, H_a \text{ is true}) =$

$= P\left(\frac{Y}{n} \ge 0.1342 \mid p = 0.10\right)$

If $p = 0.10 \qquad E\left[\frac{Y}{n}\right] = p = 0.10$

$V\left(\frac{Y}{n}\right) = \frac{p(1-p)}{n} = \frac{(0.10)(0.90)}{100} = 9\cdot10^{-4}$

type II error $= P\left(\frac{\frac{Y}{n} - 0.10}{\sqrt{9\cdot10^{-4}}} \ge \frac{0.1342 - 0.10}{0.03}\right)$

$= P(N(0,1) \ge 1.14) = 0.1271$

10.33 A random sample of 37 second-graders who participated in sports had manual dexterity scores with mean 32.19 and standard deviation 4.34. An independent sample of 37 second-graders who did not participate in sports had manual dexterity scores with mean 31.68 and standard deviation 4.56

(a) Test to see whether sufficient evidence exists to indicate that second-graders who participate in sports. Use $\alpha = 0.05$

(b) For the rejection region used in (a), calculate β when $\mu_1 - \mu_2 = 3$.

$Y_{11}, \ldots Y_{1n_1} \sim N(\mu_1, \sigma_1^2)$

$Y_{21}, \ldots Y_{2n_2} \sim N(\mu_2, \sigma_2^2)$

We test $H_0: \mu = \mu_2$ versus $H_a: \mu_1 \, \mu_2$

$\bar{Y}_1 - \bar{Y}_2$ estimates $\mu_1 - \mu_2$

$$V(\bar{Y}_1 - \bar{Y}_2) = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

Reject H_0, if

$$\bar{Y}_1 - \bar{Y}_2 \geq z_\alpha \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = 1.65\sqrt{\frac{(4.34)^2}{37} + \frac{(4.56)^2}{37}} = 1.702$$

$\bar{Y}_1 - \bar{Y}_2 = 32.19 - 31.68 = 0.51$

H_0 is not rejected

(b) type II error $= P(\bar{Y}_1 - \bar{Y}_2 \leq 1.702)$

$$\simeq P\left(N(0,1) \leq \frac{1.702 - 3}{\sqrt{\frac{(4.34)^2}{37} + \frac{(4.56)^2}{37}}}\right) = P(N(0,1) \leq -1.25) = 0.1056$$