Sampling distributions and the central limit theorem | Chapter 7

We observe a random sample $Y_1 \ldots Y_n$ from a population

$Y_1, \ldots, Y_n$ are independent identically distributed random variables

We use the data or functions of the data, to estimate or make decisions about unknown parameters of the population

Example To estimate the population mean μ, we use the sample mean $\bar{Y} = \frac{1}{n} \sum_{i=1}^{n} Y_i$

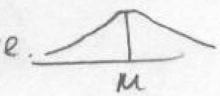
$\bar{Y}$ is a statistic.

A statistic is a function of the observable random variables in a sample and known constants.

Sampling distributions related to the normal distribution

Y is said to have a normal distribution with parameters μ and σ^2, if the density of Y is $f(Y) = \dfrac{e^{-\frac{(Y-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$

The graph of $\dfrac{e^{-\frac{(Y-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$ is a bell-shaped curve.



$f(Y) = \dfrac{e^{-\frac{(Y-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$ is a density because

(1) $f(Y) > 0$, for each y.

(2) $\int_{-\infty}^{\infty} f(Y)\,dy = 1$

Theorem For each $\mu \in \mathbb{R}, \sigma^2 > 0$, $\int_{-\infty}^{\infty} \frac{e^{-\frac{(y-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} dy = 1$

Proof By the change of variables, $\frac{y-\mu}{\sigma} = x$

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{(y-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} dy = \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

By a change to polar coordinates,

$$\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}-\frac{y^2}{2}} dx\,dy = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}} r\,dr\,d\theta$$

$$= 2\pi \int_0^{\infty} r e^{-\frac{r^2}{2}} dr = 2\pi \left(-e^{-\frac{r^2}{2}}\right)\Big|_0^{\infty} = 2\pi$$

Theorem If Y has a normal distribution with parameters μ and σ^2, then $E[Y] = \mu$ and $Var(Y) = \sigma^2$.

Proof By the change of variables $\frac{y-\mu}{\sigma} = x$ $\left(\frac{dy}{d\sigma} = dx\right.$

$$E[Y] = \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = \int_{-\infty}^{\infty} (\mu + \sigma x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx + \sigma \int_{-\infty}^{\infty} \frac{x e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \mu$$

odd function

normal density with $\mu = 0$ and $\sigma^2 = 1$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy = \int_{-\infty}^{\infty} (\mu + \sigma x)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

7-3 (740x909x24b jpeg)

$$= \mu^2 \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx + 2\mu\sigma \int_{-\infty}^{\infty} x \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx + \sigma^2 \int_{-\infty}^{\infty} x^2 \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

normal density

odd function

$$= \mu^2 + \sigma^2$$

because, by an integration by parts,

$$\int_{-\infty}^{\infty} x^2 \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = \int_{-\infty}^{\infty} \frac{(-x)}{\sqrt{2\pi}} \, d\left(e^{-\frac{x^2}{2}}\right) = \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = 1$$

Hence

$$Var(Y) = E[Y^2] - (E[Y])^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

Z is said to have a ...
and its variance is zero,

The density of a standard normal r.v Z is $f(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$

A table for the tail probabilities of a standard normal r.v. is in page 792

The table displays $P(Z \geq a)$ for $a > 0$

If $a < 0$, $P(Z > a) = 1 - P(Z > -a)$

7-4 (820x1001x24b jpeg)

Ex. 4.46 (page 173)
Use Table 4, to find the following probabilities for a standard normal r.v Z.

(a) $P(0 \le Z \le 1.2) = P(0 \le Z) - P(1.2 \le Z) = 0.5 - 0.1151 =$

$= 0.3849$

(b) $P(-0.9 \le Z \le 0) = P(0 \le Z \le 0.9) = P(0 \le Z) - P(0.9 \le Z)$

$= 0.5 - 0.1841 = 0.3159$

(c) $P(0.3 \le Z \le 1.56) = P(0.3 \le Z) - P(1.56 \le Z)$

$= 0.3821 - 0.0594 = 0.3227$

(d) $P(-0.2 \le Z \le 0.2) = 2P(0 \le Z \le 0.2) = 2(P(0 \le Z) - P(0.2 \le Z))$

$= 2(0.5 - 0.4207) = 0.1586$

(e) $P(-1.56 \le Z \le -0.2) = P(0.2 \le Z \le 1.56)$

$= P(0.2 \le Z) - P(1.56 \le Z) = 0.4207 - 0.0594 = 0.3613$

$P(-1.2 \le Z \le 2.1) = 1 - P(2.1 \le Z) - P(1.2 \le Z)$

$= 1 - 0.0179 - 0.1151 = 0.867$

7-5 (727x589x24b jpeg)

Exercise 4.47

Find the value z_0 such that

(a) $P(Z > z_0) = 0.5$ $\quad z_0 = 0$

(b) $P(Z < z_0) = 0.8643$

$1 - 0.8643 = 0.1357 = P(Z > z_0)$ $\quad z_0 = 1.1$

(c) $0.9 = P(-z_0 < Z < z_0) = 1 - 2P(Z > z_0)$

$P(Z > z_0) = \frac{1-0.9}{2} = 0.05$. So $z_0 = 1.645$

(d) $0.99 = P(-z_0 < Z < z_0) = 1 - 2P(Z > z_0)$

$P(Z > z_0) = \frac{1-0.99}{2} = 0.05$. So $z_0 = 2.575$