7-47 (828x1043x24b jpeg)

Let s^2 be the sample variance of a random sample of size 6 from the normal distribution $N(\mu, 12)$. Find $P(2.76 < S^2 < 30.80)$

$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$ $\qquad \frac{5S^2}{12} \sim \chi^2(5)$

$P(2.30 \le S^2 \le 22.5) \le P\left(\frac{5}{12}(2.76) \le \chi^2(5) \le \frac{5}{12}(30.80)\right)$

$= P(1.15 \le \chi^2(5) \le 12.8331) = 0.95 - 0.025 = 0.925$

$P(\chi^2(5) \ge 12.8325) = 0.025$

$P(\chi^2(5) \ge 1.15) = 0.95$

Let $X_1, \ldots X_5$ be a random sample of size $n=5$ from $N(0,\sigma^2)$

(a) Find the constant c so that $\frac{c(X_1-X_2)}{\sqrt{X_3^2+X_4^2+X_5^2}}$ has a t-distribution

(b) How many degrees of freedom are associated with the T?

$X_1 - X_2 \sim N(0, 2\sigma^2)$ $\qquad \frac{X_3^2+X_4^2+X_5^2}{\sigma^2} \sim \chi^2(3)$

$$\frac{\frac{X_1-X_2}{2\sigma^2}}{\sqrt{\frac{X_3^2+X_4^2+X_5^2}{3\sigma^2}}} = \frac{\sqrt{3}}{2} \frac{X_1-X_2}{\sqrt{X_3^2+X_4^2+X_5^2}} \sim t(3)$$

Extra

7-48 (807x904x24b jpeg)

Let $\bar{Y}$ be the sample mean of a random sample of size 5 from a normal distribution with mean $\mu = 0$ and $\sigma^2 = 125$. Determine c so that $P(\bar{Y} \le c) = 0.90$

$E[\bar{Y}] = \mu \quad Var(\bar{Y}) = \frac{\sigma^2}{n} = \frac{125}{5} = 25$

$0.10 = P(N(0,1) \geq 2.326)$

$\shortparallel$

$P(\bar{Y} \geq c) = P\left(\frac{\bar{Y}-\mu}{\sqrt{\sigma^2/n}} \geq \frac{c}{5}\right)$

$2.326 = \frac{c}{5} \qquad c = 5(2.326) = 11.63$

Let $X_1, \ldots X_{25}$ and let $Y_1, \ldots Y_{25}$ be two independent random samples from two normal distributions $N(0,16)$ and $N(1,9)$, respectively. Let $\bar{X}$ and let $\bar{Y}$ denote the corresponding sample means. Compute $P(\bar{X} > \bar{Y})$

$E[\bar{X}] = 0 \quad Var(\bar{X}) = \frac{16}{25} \qquad E[\bar{Y}] = 1 \quad Var(\bar{Y}) = \frac{9}{25}$

$E[\bar{X} - \bar{Y}] = -1 \quad Var(\bar{X} - \bar{Y}) = \frac{16}{25} + \frac{9}{25} = 1$

$P(\bar{X} > \bar{Y}) = P(\bar{X} - \bar{Y} > 0) = P(N(0,1) > 1) = 0.1587$

Let X_1, X_2 be a random sample from a distribution having the pdf $f(x) = e^{-x}$, $0 < x < \infty$, zero elsewhere. Show that $Z = \frac{X_1}{X_2}$ has an F distribution

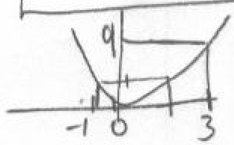
$$Y \sim \chi^2(\nu) \qquad f(y) = \frac{y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}}}{\Gamma\left(\frac{\nu}{2}\right) 2^{\nu/2}}$$

$$2X_1 \sim \chi^2(2), \quad 2X_2 \sim \chi^2(2)$$

$$\text{So} \quad Z = \frac{\frac{2X_1}{2}}{\frac{2X_2}{2}} = \frac{X_1}{X_2} \sim F(2,2)$$

If X has the pdf $f(x) = \frac{1}{4}$, $-1 < x < 3$, $f(x) = 0$ otherwise

Find the pdf of $Y = X^2$



For $0 < y < 1$, $P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} < X < \sqrt{y}) = \frac{2\sqrt{y}}{4} = \frac{\sqrt{y}}{2}$

For $1 < y <$, $P(Y < y) = P(-1 < X < \sqrt{y}) = \frac{\sqrt{y}+1}{4}$

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{\sqrt{y}}{2} & \text{if } 0 \leq y < 1 \\ \frac{\sqrt{y}+1}{4} & \text{if } 1 \leq y < 9 \\ 0 & \text{if } 9 \leq y \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}} & \text{if } 0 < y \leq 1 \\ \frac{1}{8\sqrt{y}} & \text{if } 1 < y < 9 \\ 0 & \text{else} \end{cases}$$

7-50 (756x1064x24b jpeg)

Let X_1, X_2, X_3 be three independent chi-square variables with r_1, r_2, r_3, degrees of freedom respectively

(a) Show that $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_1 + X_2$ are independent.

(b) Deduce that

$$\frac{X_1/r_1}{X_2/r_2} \quad \text{and} \quad \frac{X_3/r_3}{(X_1+X_2)/(r_1+r_2)}$$

are independent F- variables

$(Y_1, Y_2) = h(x_1, x_2)$, $\quad (x_1, x_2) = h^{-1}(y_1, y_2)$

The joint pdf of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(h^{-1}(y_1,y_2)) \cdot |J h^{-1}(y_1,y_2)|$$

$$X_1 = X_2 Y_1 = (Y_2 - X_1) Y_1 = Y_1 Y_2 - X_1 Y_1 \qquad X_1 = \frac{Y_1 Y_2}{1+Y_1} = Y_2 - \frac{Y_2}{1+Y_1}$$

$$X_2 = \frac{X_1}{Y_1} = \frac{Y_2}{1+Y_1} \qquad J = \det \begin{pmatrix} \frac{Y_2}{(1+Y_1)^2} & \frac{Y_1}{1+Y_1} \\ \frac{-Y_2}{(1+Y_1)^2} & \frac{1}{1+Y_1} \end{pmatrix}$$

$$= \frac{Y_2}{(1+Y_1)^3} + \frac{Y_1 Y_2}{(1+Y_1)^3} = \frac{Y_2}{(1+Y_1)^2}$$

$$f_{X_1,X_2}(x_1,x_2) = \frac{x_1^{\frac{r_1}{2}-1} e^{-\frac{x_1}{2}}}{\Gamma(\frac{r_1}{2})\, 2^{r_1/2}} \; \frac{x_2^{\frac{r_2}{2}-1} e^{-\frac{x_2}{2}}}{\Gamma(\frac{r_2}{2})\, 2^{\frac{r_2}{2}}}, \quad x_1, x_2 > 0$$

f_{Y_2}

7-51 (790x934x24b jpeg)

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{\left(\frac{y_1 y_2}{1+y_1}\right)^{\frac{v_1}{2}-1} e^{-\frac{y_1 y_2}{2(1+y_1)}}}{\Gamma\left(\frac{v_1}{2}\right) 2^{v_1/2}} \frac{\left(\frac{y_2}{1+y_1}\right)^{\frac{v_2}{2}-1} e^{-\frac{y_2}{2(1+y_1)}}}{\Gamma\left(\frac{v_2}{2}\right) 2^{v_2/2}} \frac{y_2}{(1+y_1)^2}$$

$$= \frac{1}{\Gamma\left(\frac{v_1}{2}\right) 2^{v_1/2} \Gamma\left(\frac{v_2}{2}\right) 2^{v_2/2}} \frac{y_1^{\frac{v_1}{2}-1}}{(1+y_1)^{\frac{v_1+v_2}{2}}} \quad y_2^{\frac{v_1+v_2}{2}-1} e^{-\frac{y_2}{2}}, \quad y_1, y_2 \geq 0$$

$Y_1 \sim F(v_1, v_2) \qquad Y_2 \sim \chi^2(v_1+v_2)$

Let X be a r.v with pdf $f(x) = 2x\, e^{-x^2}$, 0, elsewhere

Find the pdf of $Y = X^2$ $\quad x = h^{-1}(y) = \sqrt{y}$

$$f_Y(y) = f_X(h^{-1}(y)) \left|\frac{dh^{-1}(y)}{dy}\right| = 2\sqrt{y}\, e^{-y} \frac{1}{2\sqrt{y}} = e^{-y}, \quad y > 0,$$

Let X_1, X_2 be independent r.v's with pdf $f(x) = e^{-x}$, $x > 0$,
$= 0$ else

Find the pdf of $Y = X_1 + X_2$

$X_1 \sim \text{Gamma}(1,1) \qquad X_2 \sim \text{Gamma}(1,1)$

$Y = X_1 + X_2 \sim \text{Gamma}(2,1)$

$f_Y(y) = y\, e^{-y}, \quad y > 0$

7-52 (765x795x24b jpeg)

Let X_1 and X_2 have the joint pdf $f_{X_1,X_2}(x_1,x_2) = 2e^{-x_1-x_2}$, $0<x_1<x_2<\infty$, zero elsewhere. Find the joint pdf of $Y_1 = 2X_1$ and $Y_2 = X_2 - X_1$ and argue that Y_1 are independent

$$\begin{cases} Y_1 = 2X_1 \\ Y_2 = X_2 - X_1 \end{cases} \qquad \begin{cases} X_1 = Y_1/2 \\ X_2 = \frac{Y_1}{2} + Y_2 \end{cases} \qquad h^{-1}(y_1,y_2) = \left(\frac{y_1}{2}, \frac{y_1}{2} + y_2\right)$$

$$J h^{-1} = \det\begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{pmatrix} = \frac{1}{2}$$

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1X_2}\left(\frac{y_1}{2}, \frac{y_1}{2} + y_2\right)\frac{1}{2} = e^{-y_1+y_2}$$

$$0 < \frac{y_1}{2} < \frac{y_1}{2} + y_2 \qquad 0 < y_1, \ 0 < y_2$$

$f_{Y_1}(y_1) = e^{-y_1}$, $y_1 > 0$, $\quad f_{Y_2}(y_2) = e^{-y_2}$, $y_2 \geq 0$

Y_1, Y_2 are independent