7-6 (811x909x24b jpeg)

Theorem (page 297). Let Y have probability density function $f_Y(y)$. If $h(y)$ is either increasing or decreasing for all y such that $f_Y(y) > 0$, then $U = h(Y)$ has density function

$$f_U(u) = f_Y(h^{-1}(u)) \left|\frac{dh^{-1}}{du}\right|$$

Proof. Assume h is increasing

$$F_U(u) = P(U \leq u) = P(h(Y) \leq u) = P(Y \leq h^{-1}(u)) = F_Y(h^{-1}(u))$$

$$f_U(u) = f_Y(h^{-1}(u)) \, \frac{dh^{-1}(u)}{du}$$

Exercise If Y has density $f_Y(y)$, then $U = \mu + \sigma Y$, $\mu \in \mathbb{R}, \sigma > 0$ has density $f_U(u) = \frac{1}{\sigma} f\left(\frac{u-\mu}{\sigma}\right)$

$h(y) = \mu + \sigma y = u \qquad h^{-1}(u) = \frac{u-\mu}{\sigma}$

7-7 (824x1115x24b jpeg)

Theorem If X has a normal distribution, with mean μ and variance σ^2, then $\frac{X-\mu}{\sigma}$ has a standard normal distribution

Proof X has density $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Z has density $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$f_X(x) = \frac{1}{\sigma} \phi\left(\frac{x-\mu}{\sigma}\right)$

So, X has the density of $\mu + \sigma Z$

So, $\frac{X-\mu}{\sigma}$ has the distribution of Z

7-8 (765x622x24b jpeg)

Ex. 4.50 (page 173)

The weekly amount of money spent on maintenance and repairs by a company was observed, over a long period of time, to be approximately normally distributed with mean \$400 and standard deviation \$20. If \$450 is budgeted for next week, what is the probability that the actual costs will exceed the budgeted amount?

X = weekly amount of money spent on maintenance and repairs

$$P(X \geq 450) = P\left(\frac{X-400}{20} \geq \frac{450-400}{20} \right)$$

$$= P(Z \geq 2.5) = 0.0062$$

Theorem:

The mgf of a normal r.v Y with mean μ and variance σ^2 is $m_Y(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}$

Proof Since $Y = \mu + \sigma Z$, where $Z \sim N(0,1)$

$$m_Y(t) = E[e^{tY}] = E[e^{t(\mu+\sigma Z)}] = e^{t\mu}\, m_Z(t\sigma)$$

Now

$$m_Z(t) = \int_{-\infty}^{\infty} e^{tz} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}\, dz = \int_{-\infty}^{\infty} e^{\frac{t^2}{2}} \frac{e^{-\frac{(z-t)^2}{2}}}{\sqrt{2\pi}}\, dt$$

$$= e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} \underbrace{\frac{e^{-\frac{(z-t)^2}{2}}}{\sqrt{2\pi}}}_{\text{density of } N(t,1)}\, dz = e^{t^2/2}$$

So

$$m_Y(t) = e^{t\mu}\, e^{\frac{t^2\sigma^2}{2}} = e^{t\mu + \frac{t^2\sigma^2}{2}}$$