7-10 (837x1098x24b jpeg)

Theorem 6.3 (page 305) Let $Y_1 \ldots Y_n$ be independent normally distributed r.v's with $E[Y_i] = \mu_i$ and $Var(Y_i) = \sigma_i^2$ for $i = 1 \ldots n$. Let $a_1 \ldots a_n$ be constants.

Then $U = \sum_{i=1}^{n} a_i Y_i$ has a normal distribution with mean $\sum_{i=1}^{n} a_i \mu_i$ and variance $\sum_{i=1}^{n} a_i^2 \sigma_i^2$

Proof Let $m_i(t) = E[e^{tY_i}] = e^{t\mu_i + \frac{t^2\sigma_i^2}{2}}$

Then, the moment generating function of U is

$$m_U(t) = \prod_{i=1}^{n} m_i(a_i t) = \prod_{i=1}^{n} e^{t a_i \mu_i + \frac{t^2 a_i^2 \sigma_i^2}{2}} = e^{t\sum_{i=1}^{n} a_i \mu_i + \frac{t^2}{2}\sum_{i=1}^{n} a_i^2 \sigma_i^2}$$

which is the mgf of a normal distribution with mean $\sum_{i=1}^{n} a_i \mu_i$ and variance $\sum_{i=1}^{n} a_i^2 \sigma_i^2$

Theorem 7.1, page 331 Let $Y_1 \ldots Y_n$ be a random sample of size n from a normal distribution with mean μ and variance σ^2. Then $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ has a normal distribution with mean μ and variance $\frac{\sigma^2}{n}$

Proof $E[\bar{Y}] = \frac{1}{n}\sum_{i=1}^{n} E[Y_i] = \mu$, $Var(\bar{Y}) = \frac{1}{n^2}\sum_{i=1}^{n} Var(Y_i) = \frac{\sigma^2}{n}$

7.1. A forester studying the effects of fertilization on certain pine forests in the Southeast is interested in estimating the average basal area of pine trees. In studying basal areas of similar trees for many years, he has discovered these measurements (in square inches) to be normally distributed with standard deviation approximately 4 square inches. If the forester samples $n = 9$ trees, find the probability that the sample mean will be within 2 square inches of the population mean.

$$V(\bar{x}) = \frac{\sigma^2}{n} = \frac{16}{9}$$

$$P(\mu - 2 \le \bar{x} \le \mu + 2) = P(-2 \le \bar{x} - \mu \le 2)$$

$$= P\left(\frac{-2}{\frac{4}{3}} \le \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \le \frac{+2}{4/3}\right) = P\left(-\frac{3}{2} \le Z \le \frac{3}{2}\right)$$

$$= 1 - 2P\left(Z \ge \frac{3}{2}\right) = 1 - 2(0.0668) = 0.8664$$

7.2 Suppose the forester in Exercise 7.1 would like the sample mean to be within 1 square inch of the population mean, with probability 0.90. How many trees must he measure in order to ensure this degree of accuracy?

$$0.90 = P(\mu - 1 \le \bar{x} \le \mu + 1) = P\left(\frac{-1}{4/\sqrt{n}} \le \frac{\bar{x} - \mu}{4/\sqrt{n}} \le \frac{1}{4/\sqrt{n}}\right)$$

$$= 1 - 2P\left(Z \ge \frac{\sqrt{n}}{4}\right) \qquad 2P\left(Z \ge \frac{\sqrt{n}}{4}\right) = 1 - 0.90 = 0.10$$

$$P\left(Z \ge \frac{\sqrt{n}}{4}\right) = 0.05 = P(Z \ge 1.645) \qquad \frac{\sqrt{n}}{4} = 1.645$$

$$n = (4(1.645))^2 = 43.2964$$

7.62. The efficieny (in lumens per watt) of light bulbs of a certain type has population mean 9.5 and standard deviation .5, according to production specifications. The specifications for a room in which eight of these bulbs are to installed call for the average efficieny of the 8 bulbs to exceed 10. Find the probabilty that this specification for the room will be met assuming that efficieny measurements are normally distributed.

$Y_1 \ldots Y_8$ are $N(\mu = 9.5, \sigma = 0.5)$

$$\bar{Y} = \frac{1}{8}\sum_{j=1}^{8} Y_j \sim N\left(9.5, \frac{(0.5)^2}{8}\right)$$

$$P(\bar{Y} \geq 10) = P\left(\frac{\bar{Y} - 9.5}{\sqrt{\frac{(0.5)^2}{8}}} \geq \frac{10 - 9.5}{\sqrt{\frac{(0.5)^2}{8}}}\right) = P(Z \geq 2.83)$$

$= 0.0023$

7.7 Suppose that $X_1, \ldots X_m$ and $Y_1, \ldots, Y_n$ are independent random samples, with the variables X_i normally distributed with mean μ_1 and variance σ_1^2 and the variables Y_i normally distributed with mean μ_2 and variance σ_2^2. The difference between the sample means $\bar{X}-\bar{Y}$ is then a linear combination of $m+n$ normal random variables and, by Theorem 6.3, is itself normally distributed.

a. Find $E[\bar{X}-\bar{Y}]$

b. Find $V(\bar{X}-\bar{Y})$

c. Suppose that $\sigma_1^2 = 2$ and $\sigma_2^2 = 2.5$ and $m = n$. Find the sample sizes so that $\bar{X}-\bar{Y}$ will be within one unit of $\mu_1-\mu_2$ with probability 0.95

a) $E[\bar{X}-\bar{Y}] = \mu_1 - \mu_2$

(b) $V(\bar{X}-\bar{Y}) = V(\bar{X}) + V(\bar{Y}) = \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$

(c) $P(\mu_1-\mu_2-1 \le \bar{X}-\bar{Y} \le \mu_1-\mu_2+1) = 0.95$

$$= P\left(\frac{-1}{\sqrt{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{n}}} \le Z \le \frac{1}{\sqrt{\frac{\sigma_1^2}{n}+\frac{\sigma_2^2}{m}}} \right)$$

$$= P\left(\frac{-1}{\sqrt{\frac{2}{n}+\frac{2.5}{n}}} \le Z \le \frac{1}{\sqrt{\frac{2}{n}+\frac{2.5}{n}}} \right) = 1 - P\left(Z \ge \frac{\sqrt{n}}{\sqrt{4.5}}\right)$$

$\frac{\sqrt{n}}{\sqrt{4.5}} = 1.96 \qquad n = (4.5)(1.96)^2 = 17.2872$

7-14 (820x1081x24b jpeg)

7.66 page 361

From each of two normal populations with identical means and with standard deviation of 6.40 and 7.20, independent random samples of 64 observations are drawn. Find the probability that the difference between the means of the samples exceeds 0.6 in absolute value

$X_1, \ldots X_{64}$ are iid $N(\mu, \sigma = 6.4)$ $\quad \bar{X} \sim N(\mu, \frac{(6.4)^2}{64} = 0.64)$

$Y_1, \ldots Y_{64}$ are iid $N(\mu, \sigma = 7.2)$, $\quad \bar{Y} \sim N(\mu, \frac{(7.2)^2}{64} = 0.81)$

$\bar{X} - \bar{Y} \sim N(0, 1.45)$

$P(|\bar{X} - \bar{Y}| \geq 0.6) = 2P(N(0,1) \geq 0.498) = 2(0.3085) = 0.617$

Ex. 6.35 (page 307)

Let $Y_1 \dots Y_n$ be independent, normal r.v's each with mean μ and variance σ^2. Let $a_1 \dots a_n$ denote known constants.

Find the density function of the linear combination $U = \sum_{i=1}^{n} a_i Y_i$

$U \sim$ Normal $\quad E[U] = \mu \sum_{i=1}^{n} a_i \quad Var(U) = \sigma^2 \sum_{i=1}^{n} a_i^2$

$$f_U(u) = \frac{1}{\sqrt{2\pi \sigma^2 \sum_{i=1}^{n} a_i^2}} e^{-\frac{(u-\mu \sum a_i)^2}{2\sigma^2 \sum a_i^2}}$$

Ex. 5.110 Let $Y_1 \dots Y_n$ be independent r.v's with $E[Y_i] = \mu$ and $Var(Y_i) = \sigma^2$, for $i = 1, \dots n$.

Let $U_1 = \sum_{i=1}^{n} a_i Y_i$ and let $U_2 = \sum_{i=1}^{n} b_i Y_i$

Then, U_1 are U_2 are independent if and only $Cov(U_1, U_2)$

Proof The mgf of U_1 and U_2 is

$M_{U_1,U_2}(s,t) = E[e^{s \sum_{i=1}^{n} a_i Y_i + t \sum_{i=1}^{n} b_i Y_i}] = e^{sE[U_1] + tE[U_2] + Var(sU_1 + tU_2)}$

If $Cov(U_1, U_2) = 0$

$= e^{sE[U_1] + Var(sU_1)} e^{E[u U_2] + Var(sU_2)} = M_{U_1}(s) M_{U_2}(t)$

7-16 (752x900x24b jpeg)

-page 345

7.19 Let $Y_1 .. Y_5$ be a random sample of size 5 from a normal population with mean 0 and variance 1, and let $\bar{Y} = \frac{1}{5}\sum_{i=1}^{5} Y_i$

Let Y_6 be another independent observation from the same population

(a) What is the distribution of $W = \sum_{i=1}^{5} Y_i^2$?

(b) What is the distribution of $U = \sum_{i=1}^{5} (Y_i - \bar{Y})$?

(c) What is the distribution of $\sum_{i=1}^{5} (Y_i - \bar{Y})^2 + Y_6^2$?

(a) $Y_1 .. Y_5$ are iid $N(0,1)$

So $Y_1^2, \ldots Y_5^2$ are iid $X^2(1)$

and $\sum_{i=1}^{5} Y_i^2 \sim X^2(5)$

(b) $\frac{n-1}{\sigma^2} s^2 = \frac{1}{\sigma^2} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \sim X^2(n-1)$

Here, $\sigma^2 = 1$, So. $\sum_{i=1}^{5} (Y_i - \bar{Y})^2 \sim X^2(4)$

(c) $\sum_{i=1}^{5} (Y_i - \bar{Y})^2 \sim X^2(4)$, $Y_6^2 \sim X^2(1)$

$\sum_{i=1}^{5} (Y_i - \bar{Y})^2$ and Y_6^2 are independent r.v's

So, $\sum_{i=1}^{5} (Y_i - \bar{Y})^2 + Y_6^2 \sim X^2(5)$