7-25 (697x648x24b jpeg)

Example 6.4 page 289

Let Y have probability density function

$$f_Y(y) = \begin{cases} \frac{y+1}{2} & \text{if } -1 < y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the density function for $U = Y^2$

$$F_U(u) = P(U \leq u) = P(Y^2 \leq u) = P(-\sqrt{u} < Y \leq \sqrt{u}) = \int_{-\sqrt{u}}^{\sqrt{u}} \frac{y+1}{2}\,dy$$

$$= \frac{(y+1)^2}{4}\Bigg]_{-\sqrt{u}}^{\sqrt{u}} = \frac{(1+\sqrt{u})^2 - (1-\sqrt{u})^2}{4} = \sqrt{u}, \quad \text{if } 0 < u < 1$$

$$f_U(u) = \begin{cases} \frac{1}{2\sqrt{u}} & \text{if } 0 < u < 1 \\ 0 & \text{else} \end{cases}$$

7-26 (828x1090x24b jpeg)

Ex 6.1 page 291

6.1 Let Y be a random variable with probability density function given by

$$f(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the density function of $U_1 = 2Y-1$

(b) $U_2 = 1-2Y$

(c) $U_3 = Y^2$

(a) $h(y) = 2y-1 \quad h^{-1}(u) = \frac{u+1}{2}$

$$f_U(u) = f_Y\left(\frac{u+1}{2}\right)\frac{1}{2} = 2\left(1-\left(\frac{u+1}{2}\right)\right)\frac{1}{2} = \frac{1-u}{2} \quad , \text{if } -1 < u < 1$$

(b) $h(y) = 1-2y \quad h^{-1}(u) = \frac{1-u}{2}$

$$f_U(u) = f_Y\left(\frac{1-u}{2}\right)\frac{1}{2} = 2\left(1-\left(\frac{1-u}{2}\right)\right)\frac{1}{2} = \frac{1+u}{2} \quad \text{if } -1 < u < 1$$

(c) $h(y) = y^2 \quad h^{-1}(u) = \sqrt{u}$

$$f_U(u) = f_Y(\sqrt{u})\frac{1}{2\sqrt{u}} = 2(1-\sqrt{u})\frac{1}{2\sqrt{u}}) = \frac{1}{\sqrt{u}} - 1 \quad \text{if } 0 < u < 1$$

Bivariate transformation method (page 310)

Suppose that Y_1 and Y_2 are continuous random variables with joint density $f_{Y_1,Y_2}(y_1,y_2)$ and that for all (y_1,y_2) such that $f_{Y_1,Y_2}(y_1,y_2) > 0$,

$$u_1 = h_1(y_1,y_2) \quad \text{and} \quad u_2 = h_2(y_1,y_2)$$

is a one-one transformation from (y_1,y_2) to (u_1,u_2) with inverse

$$y_1 = h_1^{-1}(u_1,u_2) \quad \text{and} \quad y_2 = h_2^{-1}(u_1,u_2)$$

If $h_1^{-1}(u_1,u_2)$ and $h_2^{-1}(u_1,u_2)$ have continuous partial derivatives with respect to u_1 and u_2 and Jacobian

$$J = \det \begin{pmatrix} \dfrac{\partial h_1^{-1}}{\partial u_1} & \dfrac{\partial h_1^{-1}}{\partial u_2} \\[2ex] \dfrac{\partial h_2^{-1}}{\partial u_1} & \dfrac{\partial h_2^{-1}}{\partial u_2} \end{pmatrix} \neq 0$$

then the joint density of U_1 and U_2 is

$$\boxed{f_{U_1,U_2}(u_1,u_2) = f_{Y_1,Y_2}\left(h_1^{-1}(u_1,u_2),\, h_2^{-1}(u_1,u_2)\right) |J|}$$

Examples 6.14 (page 312-4)

Let Y_1, Y_2 be independent exponential r.v's both with mean $\beta>0$.
Find the density function of $U = \frac{Y_1}{Y_1+Y_2}$

$$f_{Y_1}(y_1) = \begin{cases} \frac{1}{\beta} e^{-y_1/\beta}, & \text{if } y_1>0 \\ 0 & \text{else} \end{cases} \qquad f_{Y_2}(y_2) = \begin{cases} \frac{1}{\beta} e^{-\frac{y_2}{\beta}} & \text{if } y_2>0 \\ 0 & \text{else} \end{cases}$$

Since Y_1 and Y_2 are independent r.v's

$$f_{Y_1 Y_2}(y_1, y_2) = \begin{cases} \frac{1}{\beta^2} e^{-\frac{y_1}{\beta} - \frac{y_2}{\beta}}, & \text{if } y_1, y_2>0 \\ 0 & \text{else} \end{cases}$$

Let $U_1 = \frac{Y_1}{Y_1+Y_2}$ $\quad U_2 = Y_1+Y_2$

$$(u_1, u_2) = h(y_1, y_2) = \left(\frac{y_1}{y_1+y_2}, y_1+y_2\right)$$

$$y_1 = u_1 u_2 \qquad y_2 = u_2 - u_1 u_2 = u_2(1-u_1)$$

$$(y_1, y_2) = h^{-1}(u_1, u_2) = (u_1 u_2, u_2(1-u_1))$$

$$J = \det\begin{pmatrix} u_2 & u_1 \\ -u_2 & 1-u_1 \end{pmatrix} = u_2 - u_1 u_2 + u_1 u_2 = u_2$$

$$f_{U_1 U_2}(u_1, u_2) = f_{Y_1 Y_2}(h^{-1}(y_1, y_2))\, |J| = \frac{1}{\beta^2} e^{-\frac{u_2}{\beta}} u_2$$

The range of (U_1, U_2) is

$$\{(u_1, u_2): 0<u_1<1, u_2>0\} \qquad U_1 \sim \text{Unif}(0,1)$$

$$f_{U_1}(u_1) = \int_0^\infty \frac{u_2}{\beta^2} e^{-\frac{u_2}{\beta}}\, du_2 = 1,$$