7-29 (837x1098x24b jpeg)

Definition

Let $Z \sim N(0,1)$ and let $W \sim X^2(\nu)$

If Z and W are independent, then

$$T = \frac{Z}{\sqrt{\frac{W}{\nu}}} \sim t(\nu)$$

$$f(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)} \frac{1}{\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}}, \quad -\infty < t < \infty$$

$$E[T] = 0 \qquad V(T) = \frac{\nu}{\nu-2}, \text{ if } \nu > 2$$

$$\left.\begin{array}{l} \frac{\sqrt{n}}{\sigma}(\bar{Y}-\mu) \sim N(0,1) \\ \frac{(n-1)S^2}{\sigma^2} \sim X^2(n-1) \end{array}\right\} t = \frac{\frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}\frac{1}{n-1}}} = \frac{\sqrt{n}(\bar{Y}-\mu)}{S} \sim t(n-1)$$

If t_α is the value such that $P(T \geq t_\alpha) = \frac{\alpha}{2}$

$$P\left(-t_{\frac{\alpha}{2}} \leq \frac{\sqrt{n}(\bar{Y}-\mu)}{S} \leq t_\alpha\right) = 1-\alpha$$

Prof Let $X \sim N(0,1)$, $Y \sim \chi^2_p$, X and Y independent

$$f_X(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \qquad f_Y(y) = \frac{y^{\frac{p}{2}-1} e^{-y/2}}{\Gamma(\frac{p}{2})\, 2^{p/2}}, \quad y > 0$$

Let $U = \dfrac{X}{\sqrt{Y/p}}$, $V = Y$ $\quad u \in \mathbb{R}, v > 0$

$x \in \mathbb{R}, y > 0$

$$X = \frac{U\sqrt{V}}{\sqrt{p}} \qquad Y = V,$$

$$f_{U,V}(u,v) = f_{X,Y}\left(x(u,v), y(u,v)\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \det\begin{pmatrix} \frac{\sqrt{v}}{\sqrt{p}} & \frac{u}{2\sqrt{v}\sqrt{p}} \\ 0 & 1 \end{pmatrix} = \frac{\sqrt{v}}{\sqrt{p}}$$

$$f_{U,V}(u,v) = \frac{e^{-\frac{u^2 v}{2p}}}{\sqrt{2\pi}} \; \frac{v^{\frac{p}{2}-1} e^{-v/2}}{\Gamma(\frac{p}{2})\, 2^{p/2}} \; \frac{\sqrt{v}}{\sqrt{p}} = \frac{e^{-\frac{u^2 v}{2p}}\, v^{\frac{p-1}{2}}\, e^{-v/2}}{\sqrt{2\pi p}\; \Gamma(\frac{p}{2})\, 2^{p/2}}$$

$$f_U(u) = \int_0^\infty \frac{e^{-\frac{u^2 v}{2p}}\, v^{\frac{p-1}{2}}\, e^{-v/2}}{\sqrt{2\pi p}\; \Gamma(\frac{p}{2})\, 2^{p/2}}\, dv = \int_0^\infty \frac{v^{\frac{p-1}{2}}\, e^{-v\left(\frac{1}{2}+\frac{u^2}{2p}\right)}}{\sqrt{2\pi p}\; \Gamma(\frac{p}{2})\, 2^{p/2}}\, dv$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\left(\frac{1}{\frac{1}{2}+\frac{u^2}{2p}}\right)^{\frac{p+1}{2}}}{\sqrt{2\pi p}\; \Gamma(\frac{p}{2})\, 2^{p/2}} = \frac{\Gamma\left(\frac{p+1}{2}\right)\left(\frac{2p}{p+u^2}\right)^{\frac{p+1}{2}}}{\sqrt{2\pi p}\; \Gamma(\frac{p}{2})\, 2^{p/2}}$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi p}\; \Gamma(\frac{p}{2})} \; \frac{1}{\left(1+\frac{u^2}{p}\right)^{\frac{p+1}{2}}}, \quad u > 0$$

7.12 Refer to Exercise 7.1. Suppose that in the forest fertilization problem of population standard deviation of basal areas is not known and must be estimated from the sample. If a random sample of $n = 9$ basal areas is to be measured, find two statistics g_1 and g_2 such that $P(g_1 \le \bar{Y} - \mu \le g_2) = 0.90$

$$\frac{\sqrt{n}(\bar{Y}-\mu)}{S} \sim t(8) \qquad P(-1.86 \le t(8) \le 1.86) = 0.90$$

$t_{0.050}$

$\downarrow$

$1.86 \leftarrow 8\,df$

$$P\left(-1.86 \le \frac{\sqrt{9}(\bar{Y}-\mu)}{S} \le 1.86\right)$$

$$P\left(-\frac{1.86}{3} S \le \bar{Y}-\mu \le \frac{1.86 S}{3}\right)$$

7-32 (837x1106x24b jpeg)

7.14 Show

(i) $E[T] = 0$ if $\nu > 1$

(ii) $V(T) = \frac{\nu}{\nu-2}$ if $\nu > 2$

(a) $E[T] = \sqrt{\nu}\, E[Z]\, E\left[\frac{1}{\sqrt{W}}\right]$

$E[Z] = 0$

$$E\left[\frac{1}{\sqrt{W}}\right] = \int_0^\infty \frac{1}{w^{1/2}} \frac{w^{\frac{\nu}{2}-1} e^{-w/2}}{\Gamma(\frac{\nu}{2}) 2^{\nu/2}} \, dw$$

$\frac{\nu}{2} - 1 - \frac{1}{2} > -1$ or $\nu > 1$

(b) $E[T^2] = \nu\, E[Z^2]\, E\left[\frac{1}{W}\right] = \nu \cdot 1 \cdot \frac{1}{\nu-2} = \frac{\nu}{\nu-2}$

$$E\left[\frac{1}{W}\right] = \int_0^\infty \frac{1}{w} \frac{w^{\frac{\nu}{2}-1} e^{-w/2}}{\Gamma(\frac{\nu}{2}) 2^{\nu/2}} \, dw = \int_0^\infty \frac{w^{\frac{\nu-2}{2}-1} e^{-w/2}}{\Gamma(\frac{\nu}{2})\, 2^{\nu/2}} \, dw$$

$$= \frac{\Gamma(\frac{\nu-2}{2})\, 2^{\frac{\nu-2}{2}}}{\Gamma(\frac{\nu}{2})\, 2^{\nu/2}} = \left(\frac{\nu-2}{2}\right) \cdot 2 = \nu - 2$$