7-33 (833x1102x24b jpeg)

Def. Let W_1 and W_2 be independent random variables, $W_1 \sim \chi^2(\nu_1)$, $W_2 \sim \chi^2(\nu_2)$. Then, $F = \dfrac{W_1/\nu_1}{W_2/\nu_2}$ is said to have an F distribution with ν_1 numerator degrees of freedom ν_1 and ν_2 denominator degrees of freedom

$$E[F] = \frac{\nu_2}{\nu_2 - 2} \quad \text{if } \nu_2 > 2$$

$$V(F) = \frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)} \quad \text{if } \nu_4 > 4$$

The density of an f distribution with degrees freedom ν_1 and ν_2 is

$$f(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{x^{\frac{\nu_1}{2}-1}}{\left(\frac{\nu_1 x}{\nu_2} + 1\right)^{\frac{\nu_1+\nu_2}{2}}}$$

7-34 (778x1085x24b jpeg)

Let $X \sim \chi^2(\nu_1)$, $Y \sim \chi^2(\nu_2)$, X, Y independent.

Let $U = \dfrac{\frac{X}{\nu_1}}{\frac{Y}{\nu_2}}$, $V = \dfrac{Y}{\nu_2}$

Then, $X = \nu_1 UV$, $Y = \nu_2 V$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \nu_1 v & \nu_1 u \\ 0 & \nu_2 \end{vmatrix} = \nu_1 \nu_2 v$$

$$f_{U,V}(u,v) = f_{X,Y}\left(x(u,v), y(u,v)\right) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

$$f_{X,Y}(x,y) = \frac{x^{\frac{\nu_1}{2}} e^{-x/2}}{\Gamma\left(\frac{\nu_1}{2}\right) 2^{\nu_1/2}} \; \frac{y^{\frac{\nu_2}{2}} e^{-y/2}}{\Gamma\left(\frac{\nu_2}{2}\right) 2^{\nu_2/2}}, \quad x, y > 0$$

$$f_{U,V}(u,v) = \frac{(\nu_1 uv)^{\frac{\nu_1}{2}} e^{-\nu_1 uv}}{\Gamma\left(\frac{\nu_1}{2}\right) 2^{\nu_1/2}} \cdot \frac{(\nu_2 v)^{\frac{\nu_2}{2}} e^{-\nu_2 v}}{\Gamma\left(\frac{\nu_2}{2}\right) 2^{\nu_2/2}} \; \nu_1 \nu_2 v$$

$$= \frac{\nu_1^{\frac{\nu_1}{2}} \nu_2^{\frac{\nu_2}{2}} u^{\frac{\nu_1}{2}-1} v^{\frac{\nu_1+\nu_2}{2}-1} e^{-v\left(\frac{\nu_1 u}{2} + \frac{\nu_2}{2}\right)}}{\Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right) 2^{\frac{\nu_1+\nu_2}{2}}}$$

$$f_U(u) = \int_0^\infty \frac{\nu_1^{\frac{\nu_1}{2}} \nu_2^{\frac{\nu_2}{2}} u^{\frac{\nu_1}{2}-1} v^{\frac{\nu_1+\nu_2}{2}-1} e^{-v\left(\frac{\nu_1 u}{2} + \frac{\nu_2}{2}\right)}}{\Gamma\left(\frac{\nu_1}{2}\right) \Gamma\left(\frac{\nu_2}{2}\right) 2^{\frac{\nu_1+\nu_2}{2}}} \, dv$$

$$= \frac{\nu_1^{\frac{\nu_1}{2}} \nu_2^{\frac{\nu_2}{2}} u^{\frac{\nu_1}{2}-1}}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right) 2^{\frac{\nu_1+\nu_2}{2}}} \Gamma\left(\frac{\nu_1+\nu_2}{2}\right) \frac{1}{\left(\frac{\nu_1 u}{2}+\frac{\nu_2}{2}\right)^{\frac{\nu_1+\nu_2}{2}}}$$

$$= \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} \frac{u^{\frac{\nu_1}{2}-1}}{\left(\frac{\nu_1}{\nu_2}u+1\right)^{\frac{\nu_1+\nu_2}{2}}}$$

7-36 (820x1106x24b jpeg)

7.13 If Y is a random variable that has an F distribution with ν_1 numerator and ν_2 denominator degrees of freedom, show that $U = \frac{1}{Y}$ has an F distribution with ν_2 numerator and ν_1 denominator degrees of freedom

$$Y \sim \frac{\frac{\chi^2(\nu_1)}{\nu_1}}{\frac{\chi^2(\nu_2)}{\nu_2}} \quad \text{So} \quad U = \frac{1}{Y} = \frac{\frac{\chi^2(\nu_2)}{\nu_2}}{\frac{\chi^2(\nu_1)}{\nu_1}}$$

7.15 Show that if T has a t distribution with ν degrees of freedom, then $U = T^2$ has an F distribution with 1 numerator degree of freedom and ν denominator degrees of freedom

$$T \sim \frac{N(0,1)}{\sqrt{\frac{\chi^2(\nu)}{\nu}}} \quad \text{So} \quad U = T^2 \sim \frac{(N(0,1))^2}{\frac{\chi^2(\nu)}{\nu}} = \frac{\frac{\chi^2(1)}{1}}{\frac{\chi^2(\nu)}{\nu}}$$

7.20 Suppose that $\underset{N(0,1)}{Y_1, \ldots Y_5}, Y_6$, $\bar{Y}$, W and U be as in exercise 7.19

$\bar{Y} = \frac{1}{5}\sum_{i=1}^{5} Y_i$, $\quad W = \sum_{i=1}^{5} Y_i^5 \sim \chi^2(5)$

$\sim N(0, \frac{1}{5})$ $\quad U = \sum_{i=1}^{5} (Y_i - \bar{Y})^2 \sim \chi^2(4)$

(a) What is the distribution of $\frac{\sqrt{5}\, Y_6}{W}$?

$t(5)$

(b) What is the distribution of $\frac{2Y_6}{\sqrt{U}}$?

$t(4)$

(c) What is the distribution of $\frac{2(5\bar{Y}^2 + Y_6^2)}{U}$?

$F(2,4)$