7-38 (752x324x24b jpeg)

The central limit theorem

Theorem: Let $Y_1, \ldots, Y_n$ be independent and identically distributed random variables with $E[Y_i] = \mu$ and $V(Y_i) = \sigma^2 < \infty$.

Then

$$\Pr\left( \frac{\sqrt{n}(\bar{Y} - \mu)}{\sigma} \le a \right) \longrightarrow P( N(0,1) \le a)$$

where $a$ is a real number

7-39 (833x1098x24b jpeg)

## A proof of the central limit theorem

**Theorem** Let $Y_n$ and $Y$ be random variables with moment generating functions $m_n(t)$ and $m(t)$, respectively. If

$$\lim_{n\to\infty} m_n(t) = m(t), \text{ for all real } t$$

Then the distribution function of $Y_n$ converges to the distribution function of $Y$, as $n \to \infty$

**CLT** Let $Y_1 \ldots Y_n$ be independent and identically distributed r.v.'s with $E[Y_i] = \mu$ and $V(Y_i) = \sigma^2 < \infty$. Then

$$\boxed{\frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma} \xrightarrow{d} N(0,1)}$$

**Proof** Let $Z_i = \frac{Y_i - \mu}{\sigma}$. Then $\frac{\sqrt{n}(\bar{Y}-\mu)}{\sigma} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i$

$$E[e^{tU_n}] = E[e^{\frac{t}{\sqrt{n}}\sum_{i=1}^{n} Z_i}] = \prod_{i=1}^{n} E[e^{\frac{tZ_i}{\sqrt{n}}}] = e^{n \log M(\frac{t}{\sqrt{n}})}$$

$$= e^{\frac{\log M(\frac{t}{\sqrt{n}})}{M(\frac{t}{\sqrt{n}})-1} \frac{M(\frac{t}{\sqrt{n}})-1}{(t/\sqrt{n})^2} t^2} \longrightarrow e^{t^2\sigma^2/2}$$

$$\lim_{x\to 1} \frac{\log x}{x-1} = 1 \quad \text{So} \quad \lim_{t\to 0} \frac{\log M(\frac{t}{\sqrt{n}})}{M(\frac{t}{\sqrt{n}})-1}$$

$$\lim_{x\to 0} \frac{M(x)-1}{x^2} = \lim_{x\to 0} \frac{M'(x)}{2x} = \lim_{x\to 0} \frac{M''(x)}{2} = \frac{\sigma^2}{2}$$

7-40 (803x1077x24b jpeg)

The normal approximation to the binomial distribution

Let $Y$ be the number of successes in $n$ trials.

$Y \sim Bin(n, p)$

Let $X_i = \begin{cases} 1 & \text{if the i-th trial results in success} \\ 0 & \text{else} \end{cases}$

$Y = \sum_{i=1}^{n} X_i \qquad E[Y] = np \qquad V(Y) = np(1-p)$

By the CLT,

$$P\left( \frac{Y - np}{\sqrt{np(1-p)}} \le a \right) \longrightarrow P(N(0,1) \le a)$$

Continuity correction

7-41 (773x980x24b jpeg)

7.30 Shear strength measurements for spot welds have been found to have standard deviation 10 psi. If 100 test welds are to be measured, what is the probability that the sample mean will be 1 psi of the true population mean

$n = 100 \quad \sigma = 10$

$P(|\bar{Y} - \mu| \leq 1) = 1 - 2P(\bar{Y} - \mu \geq 1) = 1 - 2P(N(0,1) \geq 1)$

$= 1 - 2(0.1587) = 0.6826$

7.31 Refer to Exer. 7.30. If the standard deviation of shear strength measurements for spot welds is 10 psi, how many test welds should be sampled if we want the sample mean to be within 1 psi of true mean with probability approximately 0.99

$0.99 = P(|Y - \mu| \leq 1)$

$0.005 = P(Y - \mu \geq 1) = P\left(N(0,1) \geq \frac{1}{\sqrt{\frac{10^2}{n}}}\right) = P\left(N(0,1) \geq \frac{\sqrt{n}}{10}\right)$

$\frac{\sqrt{n}}{10} = 2.576 \qquad n = (25.76)^2 = 663.57$

$\underline{664}$

7-42 (795x795x24b jpeg)

7.51 The manager of a supermarket wants to obtain information about the proportion of customer who dislike a new policy on cashing checks. How many customers should he sample if he wants the sample fraction to be within 0.15 of the true fraction with probability 0.98

Y = no. of customers who dislike a new policy

$Y \sim$ Binomial $(n= \ , p= \ )$

$0.98 = P\left( \left| \frac{Y}{n} - p \right| \le 0.15 \right)$

$0.01 = P\left( \frac{Y}{n} - p \ge 0.15 \right) = P\left( N(0,1) \ge \frac{0.15}{\sqrt{\frac{p(1-p)}{n}}} \right)$

$2.326 = \frac{0.15\sqrt{n}}{\sqrt{p(1-p}}$ $\qquad n = \frac{(2.326)^2 p(1-p)}{0.15^2} = \frac{(2.326)^2 (0.5)^2}{0.15^2}$

$= 60.11$

$n = 61$