8.2. Suppose that $E[\hat{\theta}_1] = E[\hat{\theta}_2] = \theta$, $V(\hat{\theta}_1) = \sigma_1^2$ and $V(\hat{\theta}_2) = \sigma_2^2$
A new unbiased estimator $\hat{\theta}_3$ is to be formed by

$$\hat{\theta}_3 = a\,\hat{\theta}_1 + (1-a)\,\hat{\theta}_2$$

How should the constant a be chosen in order to minimize the variance of $\hat{\theta}_3$? Assume that $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent.

$$E[\hat{\theta}_3] = \theta$$

$$V(\hat{\theta}_3) = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$$

$$\frac{d}{da}(V(\hat{\theta}_3)) = 2a\sigma_1^2 - 2(1-a)\sigma_2^2 = 2a\sigma_1^2 - 2\sigma_2^2 + 2a\sigma_2^2$$

$$a = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

$$V(\hat{\theta}_3) = \frac{\sigma_2^4\sigma_1^2}{(\sigma_1^2+\sigma_2^2)^2} + \frac{\sigma_1^4\sigma_2^2}{(\sigma_1^2+\sigma_2^2)^2} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$$

$$V(\hat{\theta}_3) \le \sigma_1^2 \qquad V(\hat{\theta}_3) \le \sigma_2^2$$

8.11 Let $Y_1 \dots Y_n$ denote a random sample of size n from a population whose density is given by

$$f(y) = \begin{cases} 3\beta^3 y^{-4} & \text{if } y \geq \beta \\ 0 & \text{else} \end{cases}$$

where $\beta > 0$ is unknown. Consider the estimator $\hat{\beta} = \min(Y_1 \dots Y_n)$

a. Derive the bias of the estimator $\hat{\beta}$

b. Derive $MSE(\hat{\beta})$

a. $F(y) = \int_\beta^y f(t)\,dt = \int_\beta^y \frac{3\beta^3}{t^4}\,dt = -\frac{\beta^3}{t^3}\Big]_\beta^y = 1 - \frac{\beta^3}{y^3}$

the density of $\hat{\beta}$ is $g_{(1)}(y) = n\left(1-F(y)\right)^{n-1} f(y) = n\left(\frac{\beta^3}{y^3}\right)^{n-1} \frac{3\beta^3}{y^4}$

$= \frac{3n\,\beta^{3n}}{y^{3n+1}}, \quad y \geq \beta$

$B(\hat{\beta}) = E[\hat{\beta}] - \beta = \int_\beta^\infty y\,\frac{3n\beta^{3n}}{y^{3n+1}}\,dy - \beta = 3n\beta^{3n}\int_\beta^\infty y^{-3n}\,dy - \beta$

$= 3n\beta^{3n}\,\frac{y^{-3n+1}}{-3n+1}\Big]_\beta^\infty = \frac{3n\beta^{3n}\beta^{1-3n}}{3n-1} - \beta = \frac{\beta}{3n-1}$

(b) $MSE(\hat{\beta}) = E[(\hat{\beta}-\beta)^2] = E[\hat{\beta}^2] - 2\beta E[\hat{\beta}] + \beta^2$

$= \frac{3n}{3n-2}\beta^2 - \frac{2\beta^2 3n}{3n-1} + \beta^2 = \frac{2\beta^2}{(3n-1)(3n-2)}$

$E[\hat{\beta}^2] = \int_\beta^\infty y^2\,\frac{3n\beta^{3n}}{y^{3n+1}}\,dy = 3n\beta^{3n}\int_\beta^\infty y^{-3n+1}\,dy = 3n\beta^{3n}\,\frac{y^{-3n+2}}{-3n+2}\Big]_\beta^\infty$

$= \frac{3n}{3n-2}\beta^2$

8.9 We have seen that, if Y has a binomial distribution with parameters n and p, then $\frac{Y}{n}$ is an unbiased estimator of p.

To estimate the variance of Y, we generally use $n\left(\frac{Y}{n}\right)\left(1-\frac{Y}{n}\right)$

a. Show that the suggested estimator is a biased estimator of $V(Y)$

b. Modify $n\frac{Y}{n}\left(1-\frac{Y}{n}\right)$ slightly to form an unbiased estimator of $V(Y)$

$E[Y] = np \qquad V(Y) = np(1-p)$

$E[Y^2] = np(1-p) + (np)^2 = np - np^2 + n^2p^2$

$E\left[n\frac{Y}{n}\left(1-\frac{Y}{n}\right)\right] = E\left[Y - \frac{Y^2}{n}\right] = np - \left(\frac{np - np^2 + n^2p^2}{n}\right)$

$= np - p + p^2 - np^2 = np(1-p) - p(1-p) = (n-1)p(1-p)$

$E\left[\frac{n}{n-1}\, n\frac{Y}{n}\left(1-\frac{Y}{n}\right)\right] = np(1-p)$

$\frac{n}{n-1}\, n\frac{Y}{n}\left(1-\frac{Y}{n}\right)$ is the requested modification

8.10 Let $Y_1, \dots Y_n$ denote a random sample of size n from a population whose density is given by

$$f(y) = \begin{cases} \dfrac{\alpha y^{\alpha-1}}{\theta^{\alpha}}, & \text{if } 0 \le y \le \theta \\ 0 & \text{elsewhere} \end{cases}$$

where $\alpha > 0$ is a known fixed value but θ is Known.

Consider the estimator $\hat{\theta} = \max(Y_1, \dots, Y_n)$

a. Show that $\hat{\theta}$ is a biased estimator for θ

$$g_{(n)}(y) = n(F(y))^{n-1} f(y) = n\left(\frac{y^{\alpha}}{\theta^{\alpha}}\right)^{n-1} \frac{\alpha y^{\alpha-1}}{\theta^{\alpha}} = n\alpha \frac{y^{\alpha n-1}}{\theta^{n\alpha}}, \quad 0 \le y \le \theta$$

$$F(y) = \frac{y^{\alpha}}{\theta^{\alpha}}$$

$$E[\hat{\theta}] = \int_0^{\theta} y \frac{n\alpha y^{\alpha n-1}}{\theta^{n\alpha}} dy = \frac{n\alpha}{\theta^{n\alpha}} \frac{y^{n\alpha+1}}{n\alpha+1}\Bigg]_0^{\theta} = \theta \frac{n\alpha}{n\alpha+1}$$

b. Find a multiple of $\hat{\theta}$ that is an unbiased estimator of θ

$$\frac{(n\alpha+1)}{n\alpha} \hat{\theta}$$

c. Derive $MSE(\hat{\theta})$

$$E[\hat{\theta}^2] = \int_0^{\theta} y^2 \frac{n\alpha y^{n\alpha-1}}{\theta^{n\alpha}} dy = \frac{n\alpha}{\theta^{n\alpha}} \frac{y^{n\alpha+2}}{n\alpha+2}\Bigg]_0^{\theta} = \frac{n\alpha}{n\alpha+2}\theta^2$$

$$V(\hat{\theta}) = \frac{n\alpha}{n\alpha+2}\theta^2 - \left(\frac{\theta\, n\alpha}{n\alpha+1}\right)^2 = \frac{n\alpha\theta^2}{(n\alpha+2)(n\alpha+1)^2}\left((n\alpha+1)^2 - n\alpha(n\alpha+2)\right)$$

$$= \frac{n\alpha\theta^2}{(n\alpha+2)(n\alpha+1)^2}$$

$$MSE(\hat{\theta}) = V(\hat{\theta}) + (E[\hat{\theta}] - \theta)^2 = \frac{n\alpha\theta^2}{(n\alpha+2)(n\alpha+1)^2} + \frac{\theta^2}{(n\alpha+1)^2} = \frac{2\theta^2}{(n\alpha+2)(n\alpha+1)}$$

$$MSE(\hat{\theta}) = E[(\hat{\theta}-\theta)^2] = E[\hat{\theta}^2 - 2\theta\hat{\theta} + \theta^2]$$