Chapter 9

Properties of point estimators and methods of estimation

Suppose that $\hat{\theta}_1$ and $\hat{\theta}_2$ are two estimators of θ

For n large, the CI's determined by $\hat{\theta}_1$ and $\hat{\theta}_2$ are

$\hat{\theta}_1 \pm z_{\frac{\alpha}{2}} \hat{\sigma}_{\hat{\theta}_1}$ and $\hat{\theta}_2 \pm z_{\frac{\alpha}{2}} \hat{\sigma}_{\hat{\theta}_2}$

The estimator with smallest variance is preferred.

Def. Given two unbiased estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ of a parameter θ, with variances $Var(\hat{\theta}_1)$ and $Var(\hat{\theta}_2)$, respectively, Then the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$ is

$$eff(\hat{\theta}_1, \hat{\theta}_2) = \frac{Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)}$$

Example 9.1

Let $Y_1, \ldots Y_n$ denote a random sample from $U(0,\theta)$.

Let $\hat{\theta}_1 = 2\bar{Y}$ and $\hat{\theta}_2 = \frac{(n+1)}{n} Y_{(n)}$

Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$

$$E[Y] = \frac{\theta}{2} \qquad Var(Y) = \frac{\theta^2}{12}$$

$$Var(2\bar{Y}) = \frac{4}{n} Var(Y) = \frac{\theta^2}{3n}.$$

$$f(y) = \frac{1}{\theta}, \ 0<y<\theta, \qquad F(y) = \frac{y}{\theta}, \ 0<y<\theta$$

$$f_{Y_{(n)}}(y) = n\left(F(y)\right)^{n-1} f(y) = n\left(\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta} = \frac{ny^{n-1}}{\theta^n}, \ 0<y<1$$

$$E[Y_{(n)}] = \int_0^\theta y \frac{ny^{n-1}}{\theta^n}\, dy = \frac{n}{n+1}\theta$$

$$E[Y_{(n)}^2] = \int_0^\theta y^2 \frac{ny^{n-1}}{\theta^n}\, dy = \frac{n}{n+2}\theta^2$$

$$Var(Y_{(n)}) = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2 = \frac{n\theta^2}{(n+2)(n+1)^2}$$

$$Var(\hat{\theta}_2) = \frac{(n+1)^2}{n^2}\frac{n\theta^2}{(n+2)(n+1)} = \frac{\theta^2}{n(n+2)}$$

$$eff(\hat{\theta}_1, \hat{\theta}_2) = \frac{Var(\hat{\theta}_2)}{Var(\theta_1)} = \frac{3}{n+2}$$

9.1 In Exercise 8.4, we considered a random sample of size 3 from an exponential distribution with pdf

$$f(y) = \begin{cases} \frac{1}{\theta} e^{-\frac{y}{\theta}} & \text{if } y \geq 0 \\ 0 & \text{else} \end{cases}$$

and determined that $\hat{\theta}_1 = Y_1$, $\hat{\theta}_2 = \frac{Y_1+Y_2}{2}$, $\hat{\theta}_3 = \frac{Y_1+2Y_2}{3}$ and $\hat{\theta}_5 = \bar{Y}$ are all unbiased estimators. Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_5$, of $\hat{\theta}_2$ relative to $\hat{\theta}_5$ and of $\hat{\theta}_3$ relative to $\hat{\theta}_5$

$Var(\hat{\theta}_1) = \theta^2$

$Var(\hat{\theta}_2) = \frac{1}{2}\theta^2$ $\quad Var(\hat{\theta}_3) = \frac{5}{9}\theta^2$ $\quad Var(\hat{\theta}_5) = \frac{\theta^2}{n}$

$eff(\hat{\theta}_1, \hat{\theta}_5) = \frac{1}{n}$

$eff(\hat{\theta}_2, \hat{\theta}_5) = \frac{2}{n}$

$eff(\hat{\theta}_3, \hat{\theta}_5) = \frac{9}{5n}$

9.3 Let $Y_1 \ldots Y_n$ denote a random sample from $U[\theta, \theta+1]$.

Let $\hat{\theta}_1 = \bar{Y} - \frac{1}{2}$ and let $\hat{\theta}_2 = Y_{(n)} - \frac{n}{n+1}$

(a) Show that both $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of θ

(b) Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$

(a) $E[Y] = \theta + \frac{1}{2}$ $\quad Var(Y) = \frac{1}{12}$

$E[\hat{\theta}_1] = \theta$ $\quad Var(\hat{\theta}_1) = \frac{1}{12n}$

$f(y) = 1$ for $\theta < y < \theta+1$, $\quad F(y) = y - \theta$, $\theta < y < \theta + 1$.

The density of $Y_{(n)}$ is

$$f_{Y_{(n)}}(y) = n(F(y))^{n-1} f(y) = n(y-\theta)^{n-1}, \quad \theta < y < \theta+1$$

$$E[Y_{(n)}] = \int_\theta^{\theta+1} y\, n(y-\theta)^{n-1}\,dx = \int_0^1 (\theta + x)\, n x^{n-1}\,dx = \theta + \frac{n}{n+1}$$

$y - \theta = x$

$$E[Y_{(n)}^2] = \int_\theta^{\theta+1} y^2 n (y-\theta)^{n-1}\,dy = \int_0^1 (\theta+x)^2 n x^{n-1}\,dx = \int_0^1 (\theta^2 n x^{n-1} + 2n\theta x^n + n x^{n+1})\,dx$$

$y - \theta = x$

$$= \theta^2 + \frac{2n\theta}{n+1} + \frac{n}{n+2}$$

$$Var(Y_{(n)}) = \theta^2 + \frac{2n\theta}{n+1} + \frac{n}{n+2} - \left(\theta + \frac{n}{n+1}\right)^2 = \frac{n}{n+2} - \frac{n^2}{(n+1)^2} = \frac{n}{(n+2)(n+1)^2}$$

$E[\hat{\theta}_2] = \theta$ $\quad Var(\hat{\theta}_2) = \frac{n}{(n+2)(n+1)^2}$

$$eff(\hat{\theta}_1, \hat{\theta}_2) = \frac{Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)} = \frac{12n^2}{(n+2)(n+1)^2}$$

n	2	3	4	7	8	9
efficiency	1.33	1.35	1.28	1.02	0.94	0.88

9.7. Suppose that $Y_1 \ldots Y_n$ denote a random sample of size n from an exponential distribution with density function given by

$$f(y) = \begin{cases} \frac{1}{\theta} e^{-y/\theta}, & \text{if } 0 < y \\ 0 & \text{elsewhere} \end{cases}$$

Show that $\hat{\theta}_1 = nY_{(1)}$ and $\hat{\theta}_2 = \bar{Y}$ are unbiased estimators.

Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$

$Y \sim Exp(\theta)$ $\quad E[Y]$ $\quad V(Y) = \theta^2$

So $E[\bar{Y}] = \theta$ $\quad V(\bar{Y}) = \frac{\theta^2}{n}$

The density of $Y_{(1)}$ is

$$g_{(1)}(y) = n(1 - F(y))^{n-1} f(y) = n(1 - (1 - e^{-y/\theta}))^{n-1} \frac{1}{\theta} e^{-y/\theta} = \frac{n e^{-\frac{ny}{\theta}}}{\theta}$$

$Y_{(1)} \sim Exp(\frac{\theta}{n})$ $\quad E[Y_{(1)}] = \frac{\theta}{n}$ $\quad V(Y_{(1)}) = \frac{\theta^2}{n^2}$

$E[\hat{\theta}_1] = \theta$ $\quad V(\hat{\theta}_1) = n^2 V(Y_{(1)}) = \theta^2$

$$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)} = \frac{\frac{\theta^2}{n}}{\theta^2} = \frac{1}{n}.$$

Example

Let $Y_1, \ldots, Y_n$ be iid r.v's with mean μ and variance σ^2.

(1) $E[\bar{Y}] = \mu$ and $Var(\bar{Y}) = \frac{\sigma^2}{n}$

(2) $E[\sum_{i=1}^n a_i Y_i] = \sum_{i=1}^n a_i \mu = \mu \iff \sum_{i=1}^n a_i = 1$

(3) If $E[\sum_{i=1}^n a_i Y_i] = \mu$, then, $\frac{\sigma^2}{n} \leq MSE(\sum_{i=1}^n a_i Y_i)$

Proof By C-S,

$$1 = \sum_{i=1}^n a_i \leq \left(\sum_{i=1}^n a_i^2\right)^{1/2} \left(\sum 1^2\right)^{1/2} = n^{1/2}\left(\sum_{i=1}^n a_i^2\right)^{1/2}$$

So

$$\frac{1}{n} \leq \sum_{i=1}^n a_i^2$$

and $\frac{\sigma^2}{n} \leq Var(\sum_{i=1}^n a_i Y_i)$