Example 9.4

Suppose that $Y_1 \ldots Y_n$ is a random sample of size n from a distribution with $E[Y_i] = \mu$ and $Var(Y_i) = \sigma^2$. Define

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2.$$ Show that $\frac{\sqrt{n}(\bar{Y}_n - \mu)}{S_n} \xrightarrow{d} N(0,1)$

By the CLT $\quad \frac{\sqrt{n}(\bar{Y}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1)$

By the LLN, $\quad \frac{1}{n} \sum_{j=1}^{n} Y_j \xrightarrow{P} E[Y]$ and $\frac{1}{n} \sum_{j=1}^{n} Y_j^2 \xrightarrow{P} E[Y^2]$

So
$$\frac{1}{n} \sum_{j=1}^{n} (Y_j - \bar{Y})^2 = \frac{1}{n} \sum_{j=1}^{n} Y_j^2 - \left(\frac{1}{n} \sum_{j=1}^{n} Y_j\right)^2 \xrightarrow{P} E[Y^2] - (E[Y])^2 = \sigma^2$$

Since $\frac{n}{n-1} \to 1$, $\quad \frac{1}{n-1} \sum_{j=1}^{n} (Y_j - \bar{Y})^2 \xrightarrow{P} \sigma^2$. So $\quad \frac{S}{\sigma} \xrightarrow{P} 1$

So $\quad \frac{\sqrt{n}(\bar{Y} - \mu)}{\sigma} \xrightarrow{d} N(0,1)$. and $\quad \frac{\sigma}{S_n} \xrightarrow{P} 1$

and $\quad \frac{\sqrt{n}(\bar{Y} - \mu)}{S} \xrightarrow{d} N(0,1)$

9.11 Suppose that $\bar{X}_1 \dots X_n$ and $Y_1 \dots Y_n$ are independent random samples from population with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Show that $\bar{X}-\bar{Y}$ is a consistent estimator of $\mu_1-\mu_2$

$$E[\bar{X}-\bar{Y}] = \mu_1-\mu_2$$

$$V(\bar{X}-\bar{Y}) = V(\bar{X}) + V(\bar{Y}) = \frac{\sigma^2}{n} + \frac{\sigma^2}{n} \longrightarrow 0.$$

So, $\bar{X}-\bar{Y}$ is a consistent estimator of $\mu_1-\mu_2$

9.12. Suppose that the population are normally distributed with $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Show that

$$\left(\sum_{i=1}^{n}(X_i-\bar{X})^2 + \sum_{i=1}^{n}(Y_i-\bar{Y})^2\right)/2n-2$$

is a consistent estimator of σ^2

Let $S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2$ and $S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\bar{Y})^2$

By Exercise 7.8 $E[S_X^2] = \sigma^2$ $V(S_X^2) = \frac{2\sigma^4}{n-1}$

$$E\left[\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2 + \sum_{i=1}^{n}(Y_i-\bar{Y})^2}{2n-2}\right] = \frac{(n-1)\sigma^2 + (n-1)\sigma^2}{2n-2} = \sigma^2$$

$$V\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2 + \sum_{i=1}^{n}(Y_i-\bar{Y})^2}{2n-2}\right) = \frac{V\left((n-1)S_X^2 + (n-1)S_Y^2\right)}{(2n-2)^2} = \frac{(n-1)^2\frac{2\sigma^4}{n-1} + (n-1)^2\frac{2\sigma^4}{n-1}}{4(n-1)^2}$$

$$\frac{4\sigma^4(n-1)}{4(n-1)^2} = \frac{\sigma^4}{n-1} \longrightarrow 0$$

Consistency

Definition The sequence of estimators $\hat{\theta}_n = \hat{\theta}_n(Y_1 \ldots Y_n)$ is said to be consistent if for each $\varepsilon > 0$, $\lim_{n \to \infty} P(|\hat{\theta}_n - \theta| \le \varepsilon) = 1$

or equivalently $\lim_{n \to \infty} P(|\hat{\theta}_n - \theta| > \varepsilon) = 0$

we denote it by $\hat{\theta}_n \xrightarrow{P} \theta$

Chebyshev's inequality

If Y is a nonnegative r.v and $a > 0$, then

$$P(Y \ge a) \le \frac{E[Y]}{a}$$

Proof $aI(Y \ge a) \le Y$, So $aP(Y \ge a) \le E[Y]$

Theorem If $E[\hat{\theta}_n] \to \theta$ and $Var(\hat{\theta}_n) \to 0$

then $\hat{\theta}_n$ is a consistent estimator of θ.

Proof By the Chebyshev's inequality, for each $\varepsilon > 0$

$$P(|\hat{\theta}_n - \theta| \ge \varepsilon) = P(|\theta_n - \theta|^2 \ge \varepsilon^2) \le \frac{E[(\hat{\theta}_n - \theta)^2]}{\varepsilon^2} = \frac{Var(\hat{\theta}_n) + (E[\hat{\theta} - \theta])^2}{\varepsilon^2} \to 0$$

9.13. Let $Y_1 \ldots Y_n$ denote a random sample from the probability density function $f(y) = \begin{cases} \theta y^{\theta-1} & \text{if } 0 < y < 1 \\ 0 & \text{else} \end{cases}$

where $\theta > 0$. Show that $\bar{Y}$ is a consistent estimator of $\frac{\theta}{\theta+1}$

$$E[\bar{Y}] = E[Y] = \int_0^1 y\,\theta y^{\theta-1}\,dy = \int_0^1 \theta y^{\theta}\,dy = \frac{\theta}{\theta+1}$$

$$V(\bar{Y}) = \frac{\sigma^2}{n} \to 0.$$

9.14. If Y has a binomial distribution with n trials and success probability p, show that $\frac{Y}{n}$ is a consistent estimator of p

$$E\left[\frac{Y}{n}\right] = p \qquad V\left(\frac{Y}{n}\right) = \frac{p(1-p)}{n} \to 0$$

Theorem:

Suppose that $\hat{\theta}_n$ converges in probability to θ and that $\hat{\theta}_n'$ converges in probability to θ'.

(a) $\hat{\theta}_n + \hat{\theta}_n'$ converges in probability to $\theta + \theta'$

(b) $\hat{\theta}_n \hat{\theta}_n'$ converges in probability to $\theta\theta'$

(c) $\dfrac{\hat{\theta}_n}{\hat{\theta}'}$ converges in probability to $\dfrac{\theta}{\theta'}$, provided that $\theta' \neq 0$

(d) If $g(\cdot)$ is a real-valued function that is continuous at θ, then $g(\hat{\theta}_n)$ converges in probability to $g(\theta)$

Theorem:

Suppose that U_n has a distribution that converges to a standard normal distribution as $n \to \infty$. If W_n converges in probability to 1 then the distribution function of $\dfrac{U_n}{W_n}$ converges to a standard normal distribution

Law of large numbers

If $Y_1, \ldots Y_n$ are iid r.v's with $E[|Y|] < \infty$, then

$$\frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{P} E[Y]$$