Rao-Blackwell theorem and minimum variance unbiased estimation

Rao-Blackwell theorem

Let $\hat{\theta}$ be a unbiased estimator for θ such that $V(\hat{\theta}) < \infty$.
If U is a sufficient statistic for, define $\hat{\theta}^* = E[\hat{\theta} \mid U]$. Then for all θ,

$$E[\hat{\theta}] = \theta \quad \text{and} \quad V(\hat{\theta}^*) \leq V(\hat{\theta})$$

Def $\hat{\theta}$ is a minimum variance unbiased estimator of θ if $\hat{\theta}$ is an unbiased estimator of θ and for any other estimator $\hat{\theta}^*$ of θ, $V(\hat{\theta}) \leq V(\hat{\theta}^*)$

Def U is a complete statistic, if for a function h,
$E_\theta[h(U)] = 0$, for each $\theta \in \Theta$, implies that $P_\theta(h(U) = 0) = 1$ for each θ.

Lehmann-Scheffé

Let $u(Y_1, \ldots, Y_n)$ be a complete sufficient statistic.
If there exists a function $h(u)$ such that $h(U)$ is an unbiased estimator of θ, then $h(U)$ is a minimum variance unbiased estimator

Example 9.6

Let $Y_1 - Y_n$ denote a random sample from a distribution with $P(Y_i = 1) = p$ and $P(Y_i = 0) = 1-p$, where p is unknown.

Find a sufficient statistic U.

Find a function of U, $h(U)$, such that $h(U)$ is an unbiased estimator of p.

$$P(Y_i = y_i) = p^{y_i}(1-p)^{1-y_i}, \quad y_i = 0, 1$$

The likelihood is

$$L(y_1 \ldots y_n \mid p) = \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i} = p^{\sum y_i}(1-p)^{n-\sum y_i} = \underbrace{\left(\frac{p}{1-p}\right)^{\sum y_i}(1-p)^n}_{g(\sum y_i,\, p)} \cdot \underbrace{1}_{h(y_1 \ldots y_n)}$$

$U = \sum_{i=1}^{n} Y_i$ is a sufficient statistic for p

$$E\left[\sum_{i=1}^{n} Y_i\right] = np$$

$\frac{1}{n}\sum_{i=1}^{n} Y_i$ is an unbiased estimator of p, which is a function of a sufficient statistic

Example 9.8 Let $Y_1 \dots Y_n$ denotes a random sample from a normal distribution with unknown mean μ and variance σ^2.

Find joint sufficient statistics for μ and σ^2.

Find a function of these joint sufficient statistics which is unbiased estimator of μ. (of σ^2).

The likelihood is

$$L(y_1 \dots y_n \mid \mu, \sigma^2) = \prod_{i=1}^{n} f(y_i \mid \mu, \sigma^2) = \prod_{i=1}^{n} \frac{e^{-\frac{(y_i-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} = \frac{e^{-\sum_{i=1}^{n}\frac{(y_i-\mu)^2}{2\sigma^2}}}{(\sqrt{2\pi})^n\,\sigma^n}$$

$$= \underbrace{\frac{e^{-\frac{\sum y_i^2}{2\sigma^2} + \frac{2\mu}{2\sigma^2}\sum y_i - \frac{n\mu^2}{2\sigma^2}}}{(\sqrt{2\pi})^n\,\sigma^n}}_{g(\sum y_i, \sum y_i^2, \mu, \sigma^2)} \cdot \underbrace{1}_{h(y_1 \dots y_n)}$$

Thus, $\sum_{i=1}^{n} Y_i$ and $\sum_{i=1}^{n} Y_i^2$, jointly, are sufficient statistics for μ and σ^2.

$\bar{Y}$ is an unbiased estimator of μ and it is a function of $\left(\sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} Y_i^2\right)$

$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\bar{Y})^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} Y_i^2 - n(\bar{Y})^2\right)$ is an unbiased estimator of σ^2 and it is a function of $\left(\sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} Y_i^2\right)$

9.49. Let $Y_1 \ldots Y_n$ denote a random sample from a normal distribution with mean μ and variance σ^2.

If μ is known and σ^2 is unknown, show that $\sum_{i=1}^{n} (Y_i-\mu)^2$ is sufficient for σ^2. Find an UMVUE of σ^2.

$$L(y_1 \ldots y_n \mid \sigma^2) = \prod_{i=1}^{n} \frac{e^{-\frac{(y_i-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma} = \frac{e^{-\sum\frac{(y_i-\mu)^2}{2\sigma^2}}}{(\sqrt{2\pi})^n \sigma^n} = \underbrace{\frac{e^{-\frac{\sum (y_i-\mu)^2}{2\sigma^2}}}{\sigma^n}}_{g\left(\sum_{i=1}^{n}(y_i-\mu)^2,\ \sigma^2\right)} \ \underbrace{\frac{1}{(\sqrt{2\pi})^n}}_{h(y_1 \ldots y_n)}$$

So, $\sum_{i=1}^{n} (Y_i-\mu)^2$ is a sufficient statistic for σ^2.

$$E\left[\frac{1}{n}\sum_{i=1}^{n} (Y_i-\mu)^2\right] = \sigma^2$$

So, $\frac{1}{n}\sum_{i=1}^{n} (Y_i-\mu)^2$ is the UMVUE of σ^2.

(Example 9.10)

9.50 Let $Y_1 \ldots Y_n$ denote a random sample from a Rayleigh distribution with parameter θ.

$$f(y|\theta) = \begin{cases} \frac{2y}{\theta} e^{-y^2/\theta}, & \text{if } y>0 \\ 0 & \text{elsewhere} \end{cases}$$

Show that $\sum_{i=1}^{n} Y_i^2$ is sufficient for θ.

Use $\sum_{i=1}^{n} Y_i^2$ to find an MVUE of θ.

$$L(y_1 \ldots y_n|\theta) = \prod_{i=1}^{n}\left(\frac{2y_i}{\theta} e^{-y_i^2/\theta}\right) = \frac{2^n \prod_{i=1}^{n} y_i \, e^{-\frac{\sum y_i^2}{\theta}}}{\theta^n}$$

$$= \underbrace{\frac{e^{-\frac{\sum y_i^2}{\theta}}}{\theta^n}}_{g(\sum_{i=1}^{n} y_i^2, \theta)} \cdot \underbrace{2^n \prod_{i=1}^{n} y_i}_{h(y_1 \ldots y_n)}$$

So, $\sum_{i=1}^{n} Y_i^2$ is a sufficient statistic for θ

$$E[Y^2] = \int_0^\infty y^2 \frac{2y}{\theta} e^{-y^2/\theta} dy = \int_0^\infty 2 \frac{(\sqrt{\theta}\sqrt{x})^3}{\theta} e^{-x} \frac{\sqrt{\theta}}{2\sqrt{x}} dx$$

$\frac{y^2}{\theta} = x \quad y = \sqrt{\theta}\sqrt{x}$

$dy = \frac{\sqrt{\theta}}{2\sqrt{x}} dx$

$$= \int_0^\infty \theta x e^{-x} dx = \theta$$

$E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i^2\right] = \theta$ The MVUE of θ is $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$

Example 9.9 Let $Y_1 \ldots Y_n$ denote a random sample from the exponential density function given by

$$f(y|\theta) = \begin{cases} \frac{1}{\theta} e^{-y/\theta} & \text{if } y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find an MVUE of $V(Y_i)$

$$L(y_1 \ldots y_n|\theta) = \prod_{i=1}^{n} \frac{e^{-y_i/\theta}}{\theta} = \frac{e^{-\sum y_i/\theta}}{\theta^n} = \underbrace{\frac{e^{-\frac{\sum y_i}{\theta}}}{\theta^n}}_{g(\sum y_i, \theta)} \cdot \underbrace{1}_{h(y_1 \ldots y_n)}$$

By the factorization theorem, $\sum_{i=1}^{n} Y_i$ is a sufficient statistic of θ

$V(Y_i) = \theta^2$

$$E\left[\left(\sum_{i=1}^{n} Y_i\right)^2\right] = V\left(\sum_{i=1}^{n} Y_i\right) + \left(E\left[\sum_{i=1}^{n} Y_i\right]\right)^2 = n\theta^2 + (n\theta)^2 = (n+n^2)\theta^2$$

$$E\left[\frac{1}{n(n+1)}\left(\sum_{i=1}^{n} Y_i\right)^2\right] = \theta^2$$

$\frac{1}{n(n+1)}\left(\sum_{i=1}^{n} Y_i\right)^2$ is the MVUE of θ^2

9.53 Let $Y_1 \ldots Y_n$ denote a random sample from the uniform distribution over the interval $(0, \theta)$. Use $Y_{(n)}$ to find an MVUE of θ.

The density of $g_{(n)}(y) = n\left(F(y)\right)^{n-1} f(y) = n\left(\frac{y}{\theta}\right)^{n-1}\frac{1}{\theta} = \frac{n y^{n-1}}{\theta^n}, \quad 0 < y < \theta$

$$E[Y_{(n)}] = \int_0^\theta y \frac{n y^{n-1}}{\theta^n}\, dy = \frac{n}{n+1}\theta$$

$\frac{(n+1) Y_{(n)}}{n}$ is an MVUE of θ.

9.51 The number of breakdowns Y per day for a certain machine is a Poisson random variable with mean λ. The daily cost of repairing these breakdowns is given by $C = 3Y^2$. If $Y_1 \ldots Y_n$ denote the observed number of breakdowns for n independently selected days, find an MVUE for $E[C]$.

$p(y|\lambda) = e^{-\lambda}\frac{\lambda^y}{y!}, \quad y = 0, 1, \ldots$

$L(y_1 \ldots y_n|\lambda) = \prod_{i=1}^n e^{-\lambda}\frac{\lambda^{y_i}}{y_i!} = e^{-n\lambda}\frac{\lambda^{\sum y_i}}{\prod y_i!}$

$\sum_{i=1}^n Y_i$ is a sufficient statistic for λ. $\quad \sum_{i=1}^n Y_i \sim \text{Poisson}(n\lambda)$

$E[\sum_{i=1}^n Y_i] = n\lambda \qquad E[(\sum_{i=1}^n Y_i)^2] = V(\sum_{i=1}^n Y_i) + (E[\sum_{i=1}^n Y_i])^2 = n\lambda + (n\lambda)^2$

$E[C] = E[3Y^2] = 3\,V(Y) + 3(E[Y])^2 = 3\lambda + 3\lambda^2 = 3\lambda(1+\lambda)$

$E[\frac{1}{n}\sum_{i=1}^n Y_i] = \lambda \qquad E[(\sum_{i=1}^n Y_i)^2 - \sum_{i=1}^n Y_i] = n^2\lambda^2$

$E[\frac{1}{n^2}(\sum_{i=1}^n Y_i)^2 - \frac{1}{n^2}\sum_{i=1}^n Y_i] = \lambda^2$

$E[3\frac{1}{n}\sum_{i=1}^n Y_i + 3\frac{1}{n^2}(\sum_{i=1}^n Y_i)^2 - \frac{3}{n^2}\sum_{i=1}^n Y_i] = 3\lambda + 3\lambda^2$

The MVUE for $E[C]$ is $3\bar{Y} + 3\bar{Y}^2 - \frac{3\bar{Y}}{n}$