The method of moments

Let $Y_1, \ldots, Y_n$ be a sample from a population with pdf $f(y \mid \theta_1, \ldots, \theta_k)$

Method of the moments estimators are found by equating the first k sample moments to the corresponding k population moments and solving the resulting system of simultaneous equations.

Let $m_k' = \frac{1}{n} \sum_{i=1}^{n} Y_i^k$ = k-th sample moment.

We solve for $\theta_1, \ldots, \theta_k$

$$m_1' = E_\theta[Y]$$

$$m_2' = E_\theta[Y^2]$$

$$\vdots$$

$$m_k' = E_\theta[Y^k]$$

We get θ_i as a function of $m_1', m_2', \ldots, m_k'$

9.62 Suppose that $Y_1, \ldots, Y_n$ constitute a random sample from a Poisson distribution with mean λ. Find the method of the moments estimator of λ

$E[Y] = \lambda$. We solve for λ the equation

$$m_1' = \lambda$$

The method of the moments estimator of λ is $m_1' = \overline{Y}$

9.66. Let $Y_1, \dots, Y_n$ denote a random sample from the probability density function,

$$f(y|\theta) = \begin{cases} (\theta+1)y^{\theta} & \text{if } 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$$

where $\theta > -1$. Find an estimator for θ by the method of moments. Show that the estimator is consistent. Is the estimator a function of the sufficient statistic $-\sum_{i=1}^{n} \ln(Y_i)$ that we obtain from the factorization criterion? What implication does this have?

$$E[Y] = \int_0^1 y(\theta+1)y^{\theta}\, dy = \frac{\theta+1}{\theta+2}$$

To find the method of the moments estimator we solve for θ in

$$m_1' = \frac{\theta+1}{\theta+2} \qquad m_1'\theta + 2m_1' = \theta+1$$

$$\hat{\theta} = \frac{1-2m_1'}{m_1'-1} = \frac{2\bar{Y}-1}{1-\bar{Y}}$$

By the law of the large numbers, $\bar{Y}$ converges to $\frac{\theta+1}{\theta+2}$ in probability

So, $\hat{\theta}$ converges to $\dfrac{2\left(\frac{\theta+1}{\theta+2}\right)-1}{1-\frac{\theta+1}{\theta+2}} = \dfrac{2\theta+2-\theta-2}{\theta+2-\theta-1} = \theta$

$$L(y_1, \dots, y_n|\theta) = \prod_{i=1}^{n} (\theta+1)y_i^{\theta} = (\theta+1)^n \left(\prod y_i\right)^{\theta} = (\theta+1)e^{\theta \sum \ln y_i}$$

$-\sum_{i=1}^{n} \ln Y_i$ is a sufficient statistic

$\hat{\theta}$ is not a function of $-\sum_{i=1}^{n} \ln Y_i$

$\hat{\theta}$ is not the MVUE, $V(\hat{\theta})$ maybe too large

9.64 If $Y_1 \ldots Y_n$ denote a random sample from the normal distribution with mean μ and variance σ^2, find the method of moments estimators of μ and σ^2?

$E[Y] = \mu \quad E[Y^2] = V(Y) + (E[Y])^2 = \sigma^2 + \mu^2$

We solve for μ and σ^2

$$\left.\begin{aligned} m_1' &= \mu \\ m_2' &= \sigma^2 + \mu^2 \end{aligned}\right\} \quad \begin{aligned} \mu &= m_1' \\ \sigma^2 &= m_2' - \mu^2 = m_2' - (m_1')^2 \end{aligned}$$

$\hat{\mu} = m_1' = \bar{Y}$

$\hat{\sigma}^2 = m_2' - (m_1')^2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right)^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$

9.65. An urn contains θ black balls and $N-\theta$ white balls. A sample of n balls is to be selected without replacement. Let Y denote the number of black balls in the sample. Show that $\frac{N}{n} Y$ is the method of the moments estimator of θ

$P(Y=y) = \dfrac{\binom{\theta}{Y}\binom{N-\theta}{n-Y}}{\binom{N}{Y}}$ $\quad Y \sim$ hypergeometric

$E[Y] = \frac{nr}{N} = \frac{n\theta}{N}$

To find the method of the moments estimator of θ, we solve for θ

$\bar{Y} = \frac{n\theta}{N} \qquad \hat{\theta} = \frac{NY}{n}$

9.66 Let $Y_1 -- Y_n$ constitute a random sample from the probability density function

$$f(y|\theta) = \begin{cases} \frac{2}{\theta^2}(\theta - y) & \text{if } 0 \le y \le \theta \\ 0 & \text{else} \end{cases}$$

(a) Find an estimator of θ by using the method of moments

(b) Is this estimator a sufficient statistic for θ?

(a) $$E[Y] = \int_0^\theta y \frac{2}{\theta^2}(\theta - y)\,dy = \int_0^\theta \left(\frac{2y}{\theta} - \frac{2y^2}{\theta^2}\right)dy = \frac{y^2}{\theta} - \frac{2y^3}{3\theta^2}\Big]_0^\theta$$

$$= \theta - \frac{2}{3}\theta = \frac{\theta}{3}.$$

Solve for θ is $\bar{Y} = \frac{\theta}{3}$ $\quad \theta = 3\bar{Y}$

$\hat{\theta} = 3\bar{Y}$ is the method of moments estimator of θ

(b) No,

9.70 Let $Y_1, \ldots, Y_n$ denote independent and identically distributed random variables from

$$f(y|\alpha) = \begin{cases} \dfrac{\alpha y^{\alpha-1}}{3^\alpha} & \text{if } 0 \le y \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

Show that $E[Y] = \dfrac{3\alpha}{\alpha+1}$, and derive the method of moments estimator for α.

$$E[Y] = \int_0^3 y \frac{\alpha y^{\alpha-1}}{3^\alpha} = \int_0^3 \frac{\alpha y^\alpha}{3^\alpha}\,dy = \frac{\alpha y^{\alpha+1}}{3^\alpha(\alpha+1)}\Bigg]_0^3 = \frac{\alpha 3^{\alpha+1}}{3^\alpha(\alpha+1)} = \frac{3\alpha}{\alpha+1}$$

The method of moments estimator for α is $\hat{\alpha}$

$$\frac{3\hat{\alpha}}{\hat{\alpha}+1} = m_1' = \bar{Y} \qquad 3\hat{\alpha} = \bar{Y}\hat{\alpha} + \bar{Y} \qquad (3-\bar{Y})\hat{\alpha} = \bar{Y}$$

$$\hat{\alpha} = \frac{\bar{Y}}{3-\bar{Y}}$$

Let $Y_1, \ldots, Y_n$ be a random sample from a Gamma distribution with parameters α and θ.

Show that $\hat{\alpha} = \dfrac{(\bar{Y})^2}{m_2' - (\bar{Y})^2}$ and $\hat{\theta} = \dfrac{m_2' - (\bar{Y})^2}{\bar{Y}}$

are the method of the moments estimators of α and θ.

Show that $\hat{\alpha}$ and $\hat{\theta}$ are consistent estimators of α and θ respectively, where $m_2' = \frac{1}{n}\sum_{j=1}^{n} Y_j^2$

If $Y \sim \text{Gamma}(\alpha, \theta)$, $E[Y] = \alpha\theta$, $\text{Var}(Y) = \alpha\theta^2$

$$E[Y^2] = \text{Var}(Y) + (E[Y])^2 = \alpha\theta^2 + (\alpha\theta)^2 = \alpha(\alpha+1)\theta^2$$

To find $\hat{\alpha}$ and $\hat{\theta}$, we solve for $\hat{\alpha}$ and $\hat{\theta}$ in

$$\left.\begin{aligned} \bar{Y} &= \alpha\theta \\ m_2' &= \alpha(\alpha+1)\theta^2 \end{aligned}\right\} \qquad \frac{m_2'}{\bar{Y}^2} = \frac{\alpha(\alpha+1)\theta^2}{(\alpha\theta)^2} = \frac{\alpha+1}{\alpha}$$

$$\alpha m_2' = (\bar{Y})^2\alpha + (\bar{Y})^2 \qquad \alpha = \frac{(\bar{Y})^2}{m_2' - (\bar{Y})^2}$$

$$\theta = \frac{\bar{Y}}{\alpha} = \frac{m_2' - (\bar{Y})^2}{\bar{Y}}$$

So $\hat{\alpha} = \dfrac{(\bar{Y})^2}{m_2' - (\bar{Y})^2}$, $\hat{\theta} = \dfrac{m_2' - (\bar{Y})^2}{\bar{Y}}$

By the LLN, $\bar{Y} \xrightarrow{P} E[Y] = \alpha\theta$, $m_2' = \frac{1}{n}\sum_{j=1}^{n} Y_j^2 \xrightarrow{P} E[Y^2] = \alpha(\alpha+1)\theta^2$

So

$$\hat{\alpha} \xrightarrow{P} \frac{(\alpha\theta)^2}{\alpha(\alpha+1)\theta^2 - (\alpha\theta)^2} = \frac{\alpha^2\theta^2}{\alpha^2\theta^2 + \alpha\theta^2 - \alpha^2\theta^2} = \alpha$$

$$\hat{\theta} \xrightarrow{P} \frac{\alpha(\alpha+1)\theta^2 - (\alpha\theta)^2}{\alpha\theta} = \frac{\alpha^2\theta^2 + \alpha\theta^2 - \alpha^2\theta^2}{\alpha\theta} = \theta$$

9.70. Let $Y_1, \ldots Y_n$ be a random sample from

$$f(y|\alpha) = \begin{cases} \frac{\alpha y^{\alpha-1}}{3^\alpha} & \text{if } 0 \le y \le 3 \\ 0 & \text{else} \end{cases}$$

where $\alpha > 1$, Derive the method of moments estimator of α.

$$E[Y] = \int_0^3 y \, \frac{\alpha y^{\alpha-1}}{3^\alpha} \, dy = \frac{\alpha \, y^{\alpha+1}}{3^\alpha(\alpha+1)} \Big]_0^3 = \frac{3\alpha}{\alpha+1}$$

To find the MME of α, we solve for α, $\quad \bar{Y} = \frac{3\alpha}{\alpha+1}$

$$\alpha\bar{Y} + \bar{Y} = 3\alpha \qquad \alpha = \frac{\bar{Y}}{3-\bar{Y}} \quad , \quad \hat{\alpha} = \frac{\bar{Y}}{3-\bar{Y}}$$

Besides, $\hat{\alpha} = \frac{\bar{Y}}{3-\bar{Y}} \xrightarrow{P} \dfrac{\frac{3\alpha}{\alpha+1}}{3 - \frac{3\alpha}{\alpha+1}} = \alpha$