Method of maximum likelihood

Suppose that the likelihood function $L(y_1 \ldots y_n | \theta_1 \ldots \theta_k)$ depends on k parameters $\theta_1 \ldots \theta_k$.

Choose as estimates, those values of the parameters that maximize the likelihood $L(y_1 \ldots y_n | \theta_1 \ldots \theta_k)$

$$L(y_1 \ldots y_n | \hat{\theta}_1 \ldots \hat{\theta}_k) = \sup_{\theta_1 \ldots \theta_k} L(y_1 \ldots y_n | \theta_1 \ldots \theta_k)$$

Invariance property of the MLE

Suppose that $\hat{\theta}$ is the mle of θ. If $t(\theta)$ is a one-to-one function of θ, then $t(\hat{\theta})$ is the mle of $t(\theta)$.

Usually, we take $\log L(y_1 \ldots y_n | \theta)$

$$\frac{\partial}{\partial \theta} \ln L(y_1 \ldots y_n | \theta) = 0.$$

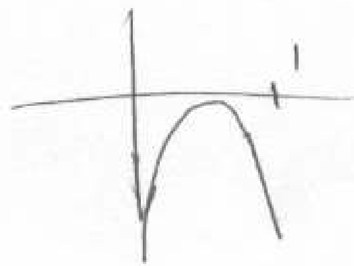
Example 9.14

A binomial experiment consisting of n trials resulted in observations $y_1 - - y_n$ where $y_i = 1$ if the i-th trial was a success, and $y_i = 0$, otherwise.

Find the maximum likelihood of p, the probability of a success.

$$f(y_i, p) = p^{y_i}(1-p)^{1-y_i}, \quad y_i = 0, 1$$

$$L(y_1 - y_n | p) = \prod_{i=1}^{n} p^{y_i}(1-p)^{1-y_i} = p^{\sum y_i}(1-p)^{n-\sum_{i=1}^{n} y_i}$$



$$\ln L(y_1 - y_n | p) = \sum y_i \ln p + (n - \sum y_i)\ln(1-p)$$

$$\frac{\partial}{\partial p} \ln L(y_1 - y_n | p) = \frac{\sum y_i}{p} - \frac{(n - \sum y_i)}{1-p} = 0$$

$$\sum y_i - p\sum y_i = np - p\sum y_i$$

$$p = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$$

9.72 Suppose that $Y_1, \ldots, Y_n$ denote a random sample from a Poisson distribution with mean λ.

a. Find the MLE $\hat{\lambda}$ of λ

b. Find the expected value and variance of $\hat{\lambda}$

c. Show that the estimator of (a) is consistent

d. What is the MLE for $P(Y=0) = e^{-\lambda}$?

a. $P(Y=y) = e^{-\lambda} \dfrac{\lambda^y}{y!}$, for $y = 0, 1, 2, \ldots$

$$L(y_1, \ldots, y_n, \lambda) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{y_i}}{y_i!} = e^{-\lambda n} \frac{\lambda^{\sum y_i}}{\prod y_i!}$$

$$\ln L(y_1, \ldots, y_n, \lambda) = -\lambda n + \left(\sum_{i=1}^{n} y_i\right) \ln \lambda - \ln\left(\prod y_i!\right)$$

$$\frac{\partial \ln L(y_1, \ldots, y_n, \lambda)}{\partial \lambda} = -n + \frac{\sum_{i=1}^{n} y_i}{\lambda} \qquad \hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} Y_i = \bar{Y}$$

$$\frac{\partial^2}{\partial \lambda^2} \ln L(y_1, \ldots, y_n, \lambda) = -\frac{\sum_{i=1}^{n} y_i}{\lambda^2}.$$

b. $E[\hat{\lambda}] = \mu = \lambda \qquad V(\hat{\lambda}) = \dfrac{\sigma^2}{n} = \dfrac{\lambda}{n}$

c. $\hat{\lambda} \to \mu = \lambda$ in probability

d. The MLE for $P(Y=0) = e^{-\lambda}$ is $e^{-\hat{\lambda}} = e^{-\bar{Y}}$

9.73. Suppose that $Y_1, \ldots Y_n$ denote a random sample from an exponential distributed population with mean θ.
Find the MLE of the population variance θ^2?

$$f(y/\theta) = \frac{1}{\theta} e^{-y/\theta}, \text{ if } y > 0$$

$$L(\theta) = \prod_{j=1}^{n} \frac{1}{\theta} e^{-y_j/\theta} = \frac{1}{\theta^n} e^{-\sum_{j=1}^{n} y_j/\theta}$$

$$\ln L(\theta) = -\frac{\sum_{j=1}^{n} y_j}{\theta} - n \ln \theta$$

$$0 = \frac{\partial}{\partial \theta} \ln L(\theta) = \frac{\sum_{j=1}^{n} y_1}{\theta^2} - \frac{n}{\theta} \qquad \theta = \bar{y}$$

$\bar{y}$ is the MLE of θ

$(\bar{y})^2$ is the MLE of θ^2

9.74 Let $Y_1 \ldots Y_n$ denote a random sample from the density function given by

$$f(y|\theta) = \begin{cases} \dfrac{r y^{r-1}}{\theta} e^{-y^r/\theta}, & \text{if } \theta > 0, y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

where r is a known positive constant.

a. Find a sufficient statistic for θ.

b. Find the maximum-likelihood estimator of θ.

c. Is the estimator in part (b) an MVUE for θ?

a. $$L(y_1 \ldots y_n|\theta) = \prod_{i=1}^{n} \frac{r y_i^{r-1}}{\theta} e^{-y_i^r/\theta} = \frac{r^n \prod_{i=1}^{n} y_i^{r-1}}{\theta^n} e^{-\sum y_i^r/\theta}$$

$$\underbrace{\frac{e^{-\sum y_i^r/\theta}}{\theta^n}}_{g(\sum_{i=1}^{n} y_i^r, \theta)} \quad \underbrace{r^n \prod_{i=1}^{n} y_i^{r-1}}_{h(y_1 \ldots y_n)}$$

$\sum_{i=1}^{n} y_i^r$ is a sufficient statistic for θ

b. $$\ln L(y_1 \ldots y_n|\theta) = -\sum_{i=1}^{n} \frac{y_i^r}{\theta} - n \ln\theta + \ln\left(r^n \prod_{i=1}^{n} y_i^{r-1}\right)$$

$$\frac{\partial}{\partial\theta} \ln L(y_1 \ldots y_n|\theta) = \frac{\sum_{i=1}^{n} y_i^r}{\theta^2} - \frac{n}{\theta} = 0 \qquad \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} y_i^r$$

c. Yes, the MLE $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} y_i^r$ is the MVUE for θ