Some large sample properties of MLE's

If $\hat{\theta}$ is the MLE of θ, then

(1) $\dfrac{\hat{\theta}-\theta}{\sigma(\hat{\theta})} \xrightarrow{d} N(0,1)$

(2) $\dfrac{\sigma(\hat{\theta})}{\sqrt{n E[-\frac{\partial^2}{\partial\theta^2}\ln f(X,\theta)]\big|_{\theta=\hat{\theta}}}} \xrightarrow{P} N(0,1)$

(3) $\sqrt{n E[-\frac{\partial^2}{\partial\theta^2}\ln f(X,\theta)]\big|_{\theta=\hat{\theta}}}\,(\hat{\theta}-\theta) \xrightarrow{d} N(0,1)$

(4) $\hat{\theta} \pm z_{\frac{\alpha}{2}} \dfrac{1}{\sqrt{n E[-\frac{\partial^2}{\partial\theta^2}\ln f(X,\theta)]\big|_{\theta=\hat{\theta}}}}$ is an asymptotic

$100(1-\alpha)$ confidence interval for θ

(5) If t is a differentiable function

$\sqrt{n E[-\frac{\partial^2}{\partial\theta^2}\ln f(X,\theta)]\big|_{\theta=\hat{\theta}}}\left(\dfrac{t(\hat{\theta})-t(\theta)}{t'(\hat{\theta})}\right) \xrightarrow{d} N(0,1)$

and

$t(\hat{\theta}) \pm z_{\frac{\alpha}{2}} \dfrac{|t'(\hat{\theta})|}{\sqrt{n E[-\frac{\partial^2}{\partial\theta^2}\ln f(X,\theta)]\big|_{\theta=\hat{\theta}}}}$ is an asymptotic

$100(1-\alpha)\%$ confidence interval for θ.

Example 9.18

Let $Y_1, \ldots, Y_n$ denote a random sample of size n from a Bernoulli distribution.

(a) Find the MLE of p

(b) Find $E\left[-\frac{\partial^2}{\partial p^2} \ln f(Y, p)\right]$

(c) Find an asymptotic $100(1-\alpha)\%$ confidence interval for p

(d) Find an asymptotic $100(1-\alpha)\%$ confidence interval for $Var(Y) = p(1-p)$

(a) $f(y,p) = p^y (1-p)^{1-y}, \quad y = 0, 1$

$$L(p) = \prod_{j=1}^{n} f(y_j, p) = \prod_{j=1}^{n} p^{y_j}(1-p)^{1-y_j} = p^{n\bar{y}} (1-p)^{n-n\bar{y}}$$

$$\ln L(p) = n\bar{y} \ln p + n(1-\bar{y}) \ln(1-p)$$

$$0 = \frac{\partial}{\partial p} \ln L(p) = \frac{n\bar{y}}{p} - \frac{n(1-\bar{y})}{1-p}$$

$n\bar{y}(1-p) = n\bar{y} - n\bar{y}p$

$\parallel$

$pn(1-\bar{y}) = np - np\bar{y}$

So $p = \bar{y}$

The MLE for p is $\hat{p} = \bar{y}$

(b) $\ln f(y,p) = y \ln p + (1-y)\ln(1-p)$

$$\frac{\partial}{\partial p}\ln f(y,p) = \frac{y}{p} - \frac{(1-y)}{1-p}$$

$$\frac{\partial^2}{\partial p^2}\ln f(y,p) = -\frac{y}{p^2} - \frac{(1-y)}{(1-p)^2}$$

$$E\left[-\frac{\partial}{\partial p^2}\ln f(y,p)\right] = E\left[\frac{y}{p^2} + \frac{1-y}{(1-p)^2}\right] = \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}$$

(c) A $100(1-\alpha)\%$ confidence interval for p is

$$\hat{p} \pm z_{\frac{\alpha}{2}} \frac{1}{\sqrt{nE\left[-\frac{\partial^2}{\partial p^2}\ln f(X,p)\right]\Big|_{p=\hat{p}}}}$$

$$= \hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

(d) $t(p) = p(1-p) = p - p^2 \qquad t'(p) = 1-2p$

A $100(1-\alpha)\%$ confidence interval for $t(p) = p(1-p)$ is

$$\hat{p} \pm z_{\frac{\alpha}{2}} \frac{|t'(\hat{p})|}{\sqrt{nE\left[-\frac{\partial^2}{\partial p^2}\ln f(X,p)\right]\Big|_{p=\hat{p}}}} = \hat{p} \pm z_{\frac{\alpha}{2}} \frac{|1-2\hat{p}|\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}$$

$$= \hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})(1-2\hat{p})^2}{n}}$$

9.90. Suppose that $Y_1, \dots Y_n$ constitute a random sample of size n from an exponential distribution with mean θ. Find a $100(1-\alpha)\%$ confidence interval for $t(\theta) = \theta^2$

The confidence interval is

$$t(\hat{\theta}) \pm z_{\frac{\alpha}{2}} \frac{|t'(\hat{\theta})|}{\sqrt{n E[-\frac{\partial^2}{\partial\theta^2} \ln f(x,\theta)]\Big|_{\theta=\hat{\theta}}}}$$

where $\hat{\theta}$ is the MLE of θ.

We have

$$f(y|\theta) = \frac{1}{\theta} e^{-y/\theta}, \ \forall\, y > \theta$$

$$L(\theta) = \prod_{j=1}^{n} \frac{1}{\theta} e^{-y_j/\theta} = \frac{1}{\theta^n} e^{-n\bar{Y}/\theta}$$

$$\ln L(\theta) = -n \ln\theta - \frac{n\bar{Y}}{\theta}$$

$$0 = \frac{\partial}{\partial\theta} \ln L(\theta) = \frac{-n}{\theta} + \frac{n\bar{Y}}{\theta^2} \qquad \hat{\theta} = \bar{Y} \text{ is the MLE of } \theta$$

$$\ln f(y,\theta) = -\ln\theta - \frac{y}{\theta} \qquad \frac{\partial}{\partial\theta} \ln f(y|\theta) = -\frac{1}{\theta} + \frac{y}{\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2} \ln f(y|\theta) = \frac{1}{\theta^2} - \frac{2y}{\theta^3}$$

$$E[-\frac{\partial^2}{\partial\theta^2} \ln f(y|\theta)] = E[\frac{2y}{\theta^3} - \frac{1}{\theta^2}] = \frac{2\theta}{\theta^3} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

$$t'(\theta) = 2\theta$$

A $100(1-\alpha)\%$ confidence interval for θ^2 is

$$(\bar{y}) \pm z_{\frac{\alpha}{2}} \frac{|2\bar{y}|}{\sqrt{n \frac{1}{(\bar{y})^2}}} = \bar{y} \pm z_{\frac{\alpha}{2}} \frac{2}{\sqrt{n}} (\bar{y})^2$$

<u>Example</u> Let $Y_1, \ldots, Y_n$ be a random sample from a normal distribution with mean 0 and variance θ

Find an approximate large sample $100(1-\alpha)\%$ confidence interval for θ

$$f(y|\theta) = \frac{e^{-y^2/2\theta}}{\sqrt{2\pi}\sqrt{\theta}}$$

$$L(\theta) = \prod_{i=1}^{n} \frac{e^{-y_i^2/2\theta}}{\sqrt{2\pi}\sqrt{\theta}} = \frac{e^{-\frac{\sum y_i^2}{2\theta}}}{(\sqrt{2\pi})^n \theta^{n/2}}$$

$$\ln L(\theta) = -\frac{\sum y_i^2}{2\theta} - \frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\theta$$

$$\frac{\partial}{\partial\theta}\ln L(\theta) = \frac{\sum y_i^2}{2\theta^2} - \frac{n}{2\theta} = 0 \qquad \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} y_i^2 \text{ is the MLE of } \theta.$$

$$\sigma^2(t(\theta)) = \frac{\left(\frac{\partial t(\theta)}{\partial\theta}\right)^2}{nE\left[-\frac{\partial^2}{\partial\theta^2}\ln f(X,\theta)\right]}$$

$$t(\theta) = \theta \qquad \frac{\partial t(\theta)}{\partial\theta} = 1$$

$$\ln f(y|\theta) = \frac{-y^2}{2\theta} - \frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\theta$$

$$\frac{\partial}{\partial\theta}\ln f(y|\theta) = \frac{y^2}{2\theta^2} - \frac{1}{2\theta}$$

$$\frac{\partial^2}{\partial\theta^2}\ln f(y|\theta) = \frac{-y^2}{\theta^3} + \frac{1}{2\theta^2}$$

$$E\left[-\frac{\partial^2}{\partial\theta^2}\ln p(Y\mid\theta)\right] = E\left[\frac{Y^2}{\theta^3}-\frac{1}{2\theta^2}\right] = \frac{\theta}{\theta^3}-\frac{1}{2\theta^2} = \frac{1}{2\theta^2}$$

$$\sigma^2(t(\theta)) = \frac{1}{n\frac{1}{2\theta^2}} = \frac{2\theta^2}{n}$$

$$\hat{\sigma}(t(\theta)) = \frac{\sqrt{2}\,\hat{\theta}}{\sqrt{n}}$$

An approximate large sample $1-\alpha$ confidence interval for θ

is

$$\hat{\theta} \pm z_{\frac{\alpha}{2}}\frac{\sqrt{2}\,\hat{\theta}}{\sqrt{n}}$$