1 Review of exponential, gamma, chi-square distribution

The gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt, \alpha > 0.$$

Theorem 1.1. *The gamma function satisfies the following properties:*
(a) For each $\alpha > 1$, $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$.
(b) For each integer $n \geq 1$, $\Gamma(n) = (n-1)!$.
(c) $\Gamma(1/2) = \sqrt{\pi}$.

Proof. For each $\alpha > 1$, by an integration by parts

$$\begin{aligned}&\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt = \int_0^\infty t^{\alpha-1}d(-e^{-t})\,dt = t^{\alpha-1}(-e^{-t})\Big|_0^\infty - \int_0^\infty(-e^{-t})d(t^{\alpha-1})\\ &= t^{\alpha-1}(-e^{-t})\Big|_0^\infty + \int_0^\infty(\alpha-1)t^{\alpha-2}e^{-t}\,dt = (\alpha-1)\Gamma(\alpha-1).\end{aligned}$$

Next, we prove by induction that for each integer $n \geq 1$, $\Gamma(n) = (n-1)$. The case $n = 1$ holds because

$$\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1.$$

The case n implies the case $n+1$, because if $\Gamma(n) = (n-1)!$, then $\Gamma(n+1) = n\Gamma(n) = n(n-1)! = n!$.

Finally, by the change of variables $t = x^2/2$, $(dt = x\,dx)$,

$$\begin{aligned}&\Gamma(1/2) = \int_0^\infty t^{-1/2}e^{-t}\,dt = \int_0^\infty \sqrt{\tfrac{2}{x^2}}e^{-\frac{x^2}{2}}x\,dx = \sqrt{2}\int_0^\infty e^{-\frac{x^2}{2}}\,dx\\ &= \tfrac{\sqrt{2}}{2}\int_{-\infty}^\infty e^{-\frac{x^2}{2}}\,dx = \tfrac{\sqrt{2}}{2}\sqrt{2\pi} = \sqrt{\pi}\end{aligned}$$

□

In particular, we have that

$$\int_0^\infty x^n e^{-x}\,dx = \Gamma(n+1) = n! \tag{1}$$

By the change of variables $y = \frac{x}{\theta}$,

$$\int_0^\infty x^{\alpha-1}e^{-x/\theta}\,dx = \int_0^\infty x^{\alpha-1}\theta^\alpha e^{-y}\,dy = \theta^\alpha\Gamma(\alpha) \tag{2}$$

Exercise 1.1. *Find:*
(i) $\int_0^\infty x^3e^{-x}\,dx$.
(ii) $\int_0^\infty x^{12}e^{-x}\,dx$.
(iii) $\int_0^\infty x^{23}e^{-2x}\,dx$.
(iv) $\int_0^\infty x^{24}e^{-x/3}\,dx$.

Exercise 1.2. *Use integration by parts to show that*

$$\int xe^{-x}\,dx = -e^{-x}(1+x)+c.$$

Exercise 1.3. *Use integration by parts to show that*

$$\int \frac{x^2}{2}e^{-x}\,dx = -e^{-x}(1+x+\frac{x^2}{2})+c.$$

Exercise 1.4. *Prove that for each integer $n \geq 1$,*

$$\int \frac{x^n}{n!}e^{-x}\,dx = -\sum_{j=0}^{n} e^{-x}\frac{x^j}{j!}+c.$$

Hint: use integration by parts and induction.

Definition 1.1. *A r.v. X is said to have an exponential distribution with parameter $\lambda > 0$, if the density of X is given by*

$$f(x) = \begin{cases} \frac{e^{-\frac{x}{\lambda}}}{\lambda} & \text{if } \ x \geq 0 \\ 0 & \text{if } \ x < 0 \end{cases}$$

We denote this by $X \sim \mathrm{Exponential}(\lambda)$.

The above function f defines a bona fide density because it is nonnegative and

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_0^{\infty} \frac{e^{-\frac{t}{\lambda}}}{\lambda}\,dt = -e^{-\frac{t}{\lambda}}\Big|_0^{\infty} = 1.$$

Theorem 1.2. *Let X be a r.v. with an exponential distribution and parameter $\lambda > 0$, then*

$$E[X] = \lambda, \mathrm{Var}(X) = \lambda^2, E[X^k] = \lambda^k k!, M(t) = \frac{1}{1-\lambda t}, \text{ if } t < \lambda^{-1}.$$

Proof. Using (1.2),

$$E[X^k] = \textstyle\int_0^{\infty} x^k \frac{e^{-\frac{x}{\lambda}}}{\lambda}\,dx = \frac{1}{\lambda}\Gamma(k+1)\lambda^{k+1} = k!\lambda^k$$

In particular,

$$\begin{aligned} &E[X] = \lambda, E[X^2] = 2\lambda^2 \\ &\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \lambda^2. \end{aligned}$$

We have that for $t < \lambda^{-1}$,

$$M(t) = E[e^{tX}] = \int_0^{\infty} e^{tx}\frac{e^{-\frac{x}{\lambda}}}{\lambda}\,dx = \frac{1}{\lambda}\int_0^{\infty} e^{-x(\frac{1-\lambda t}{\lambda})}\,dx = \frac{1}{\lambda}\frac{\lambda}{1-\lambda t} = \frac{1}{1-\lambda t}.$$

$\square$

The cumulative distribution function of an exponential distribution with mean $\lambda > 0$ is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = 1 - e^{-\frac{x}{\lambda}},\ x \ge 0.$$

The exponential distribution satisfies that for each $s, t \ge 0$,

$$\mathbb{P}(X > s + t | X > t) = P(X > s).$$

This is property is called the memoryless property of the exponential distribution.

Definition 1.2. *X has a gamma distribution with parameters $\alpha > 0$ and $\theta > 0$, if the density of X is*

$$f(x) = \begin{cases} \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\theta^\alpha \Gamma(\alpha)} & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases}$$

We denote this by $X \sim \text{Gamma}(\alpha, \theta)$.

The above function f defines a bona fide density because, by (1.2),

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^\infty \frac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\Gamma(\alpha)\theta^\alpha}\,dx = 1.$$

A gamma distribution with parameter $\alpha = 1$ is an exponential distribution.

Theorem 1.3. *If X has a gamma distribution with parameters α and θ, then*

$$E[X] = \alpha\theta, \text{Var}(X) = \alpha\theta^2, E[X^k] = \frac{\Gamma(\alpha+k)\theta^k}{\Gamma(\alpha)}, \text{Var}(X) = \alpha\theta^2, M(t) = \frac{1}{(1-\theta t)^\alpha}, \text{ if } t < \frac{1}{\theta}.$$

Proof. Using (1.2),

$$E[X^k] = \int_0^\infty x^k \tfrac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\Gamma(\alpha)\theta^\alpha}\,dx = \tfrac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty x^{k+\alpha-1}e^{-\frac{x}{\theta}}\,dx = \tfrac{1}{\Gamma(\alpha)\theta^\alpha}\Gamma(k+\alpha)\theta^{k+\alpha} = \tfrac{\Gamma(\alpha+k)\theta^\alpha}{\Gamma(\alpha)}.$$

In particular,

$$\begin{aligned} &E[X] = \tfrac{\Gamma(\alpha+1)\theta^1}{\Gamma(\alpha)} = \alpha\theta, E[X^2] = \tfrac{\Gamma(\alpha+2)\theta^2}{\Gamma(\alpha)} = (\alpha+1)\alpha\theta^2, \\ &\text{Var}(X) = E[X^2] - (E[X])^2 = \alpha\theta^2. \end{aligned}$$

We have that for $t < \frac{1}{\theta}$,

$$\begin{aligned} &M(t) = E[e^{tX}] = \int_0^\infty e^{tx}\tfrac{x^{\alpha-1}e^{-\frac{x}{\theta}}}{\Gamma(\alpha)\theta^\alpha}\,dx = \tfrac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty x^{\alpha-1}e^{-x\left(\frac{1-\theta t}{\theta}\right)}\,dx \\ &= \tfrac{1}{\Gamma(\alpha)\theta^\alpha}\left(\tfrac{\theta}{1-\theta t}\right)^\alpha \Gamma(\alpha) = \tfrac{1}{(1-\theta t)^\alpha}. \end{aligned}$$

$\square$

Definition 1.3. *Given a positive integer ν, a random variable X is said to have a chi-square distribution with degrees of freedom ν if and only if X has a gamma distribution with parameters $\alpha = \nu/2$, and $\beta = 2$, i.e. X has a chi-square distribution with degrees of freedom ν if its density is*

$$f(x) = \begin{cases} \frac{x^{\frac{\nu}{2}-1}e^{-\frac{x}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

We denote this by $X \sim \chi^{(\nu)}$.

Theorem 1.4. *If X has a chi-square distribution with degrees of freedom ν, then*

$$E[X] = \nu, \mathrm{Var}(X) = 2\nu, M(t) = \frac{1}{(1-2t)^{\nu/2}}, \text{ if } t < \frac{1}{2}.$$

Theorem 1.5. *Suppose that X and Y are two independent r.v.'s with chi-square distributions with respective degrees of freedom ν_1 and ν_2, then $X+Y$ has a chi-square distribution with $\nu_1 + \nu_2$ degrees of freedom.*

Proof. Suppose that X and Y have respective mgf's $M_X(t) = \frac{1}{(1-2t)^{\nu_1/2}}$ and $M_Y(t) = \frac{1}{(1-2t)^{\nu_2/2}}$. Hence, that $X+Y$ has mgf $M_{X+Y}(t) = \frac{1}{(1-2t)^{(\nu_1+\mu_2)/2}}$, which is the mgf of a chi-square distribution with $\nu_1 + \nu_2$ degrees of freedom. □

Theorem 1.6. *Suppose that Z has a standard normal distribution, then $Y = Z^2$ has a chi-square distribution with one degree of freedom.*

Proof. The cdf of $Y = Z^2$ is

$$F_Y(y) = \mathbb{P}[Z^2 \leq y] = \mathbb{P}[-\sqrt{y} \leq Z \leq \sqrt{y}] = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}}\,dz = 2\int_0^{\sqrt{y}} \frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}}\,dz.$$

Hence, the pdf of Y is

$$f_Y(y) = 2\frac{e^{\frac{-y}{2}}}{\sqrt{2\pi}}\frac{d}{dy}(\sqrt{y}) = 2\frac{e^{\frac{-y}{2}}}{\sqrt{2\pi}}\frac{1}{2\sqrt{y}} = \frac{y^{\frac{1}{2}-1}e^{-\frac{y}{2}}}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})}.$$

□

Corollary 1.1. *Suppose that $Z_1, \ldots, Z_n$ are independent identically distributed r.v.'s with a standard normal distribution, then $Z_1^2 + \cdots + Z_n^2$ has a chi-square distribution with n degrees of freedom.*