2nd homework

1. Let U and let V be two independent r.v's. $U \sim \chi^2(\nu_1)$, $V \sim \chi^2(\nu_2)$

Then, F has the distribution of $\dfrac{\frac{U}{\nu_1}}{\frac{V}{\nu_2}} = \dfrac{U\nu_2}{\nu_1 V}$

So

$$E[F] = E\left[\frac{U\nu_2}{\nu_1 V}\right] = \frac{\nu_2}{\nu_1} E[U]E\left[\frac{1}{V}\right] = \frac{\nu_2}{\nu_1}\, \nu_1 \frac{1}{\nu_2 - 2} = \frac{\nu_2}{\nu_2-2}$$

because

$$E\left[\frac{1}{V}\right] = \int_0^\infty \frac{1}{y} \frac{y^{\frac{\nu_2}{2}-1} e^{-\frac{y}{2}}}{\Gamma(\frac{\nu_2}{2})\, 2^{\nu_2/2}}\, dy = \int_0^\infty \frac{y^{\frac{\nu_2-2}{2}-1} e^{-\frac{y}{2}}}{\Gamma(\frac{\nu_2}{2})\, 2^{\frac{\nu_2}{2}}}\, dy$$

$$= \frac{\Gamma\left(\frac{\nu_2-2}{2}\right) 2^{\frac{\nu_2-2}{2}}}{\Gamma\left(\frac{\nu_2}{2}\right) 2^{\frac{\nu_2}{2}}} = \frac{1}{\left(\frac{\nu_2}{2}-1\right) \quad 2} = \frac{1}{2\nu_2 - 2}$$

2. In parts (a), (b), we use $f_U(u) = f_Y(h^{-1}(u)) \left|\dfrac{d h^{-1}(u)}{du}\right|$

(a) $h(y) = 5 - 3y = u \qquad y = \dfrac{5-u}{3} = h^{-1}(u)$

$$f_U(u) = \frac{3}{2}\left(\frac{5-u}{3}\right)^2 \frac{1}{3} = \frac{(5-u)^2}{18} \quad \text{if} \quad 2 < u < 8$$

(b) $h(y) = 5 - e^y \qquad y = \ln(5-u)$

$$f_U(u) = \frac{3}{2}\left(\ln(5-u)\right)^2 \frac{1}{5-u} \qquad \text{if} \quad 5 - e < u < 5 - e^{-1}$$

(c) $U = 3Y^2 - 4$

$$F_U(u) = P(3Y^2 - 4 \le u) = P\left(Y^2 \le \frac{u+4}{3}\right) = P\left(-\sqrt{\frac{u+4}{3}} \le Y \le \sqrt{\frac{u+4}{3}}\right)$$

$$= \int_{-\sqrt{\frac{u+4}{3}}}^{\sqrt{\frac{u+4}{3}}} \frac{3y^2}{2}\,dy = \frac{y^3}{2}\Bigg]_{-\sqrt{\frac{u+4}{3}}}^{\sqrt{\frac{u+4}{3}}} = \left(\frac{u+4}{3}\right)^{3/2} = \frac{(u+4)^{3/2}}{3\sqrt{3}}$$

for $-4 < u < -1$

$$f_u(u) = \begin{cases} \dfrac{(u+4)^{1/2}}{2\sqrt{3}} & \text{if } -4 < u < 1 \\ 0 & \text{else} \end{cases}$$

3. $U = Y^{1/m}$ $\quad h(y) = y^{1/m}$ $\quad h^{-1}(u) = u^m$

$$f_u(u) = f_y(u^m)\, m\, u^{m-1} = \frac{1}{B} \cdot e^{-\frac{u^m}{B}}\, m\, u^{m-1} \sim \text{Weib}\,(m, \alpha)$$

4.

$$(u_1, u_2) = h(y_1, y_2) = \left(y_1, \frac{y_2}{y_1}\right)$$

$$y_1 = u_1 \qquad y_2 = u_1 u_2$$

$$\left|\frac{\partial h(y_1, y_2)}{\partial}\right| = \begin{vmatrix} 1 & 0 \\ u_2 & u_1 \end{vmatrix} = u_1$$

$$f_{u_1, u_2}(u_1, u_2) = \frac{u_1 e^{-\frac{u_1(1+u_2)}{2}}}{8} u_1, \qquad 0 < u_1,\ 0 < u_2$$

$$f_{u_2}(u_2) = \int_0^\infty \frac{u_1^2 e^{-u_1\left(\frac{1+u_2}{2}\right)}}{8} du_1 = \frac{\Gamma(3)\left(\frac{2}{1+u_2}\right)^3}{8} = \frac{2}{(1+u_2)^3}, \quad u_2 > 0.$$

5.

(a) $1/2$

(b)

x	-100	0	100	200
$P(X=x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$$E[X] = (-100)\frac{1}{4} + (0)\frac{1}{4} + (100)\frac{1}{4} + (200)\left(\frac{1}{4}\right) = 50$$

$$E[X^2] = (-100)^2\frac{1}{4} + (0)^2\frac{1}{4} + (100)^2\frac{1}{4} + (200)^2\frac{1}{4} = 15000$$

$$E\left[\sum_1^{100} X_i\right] = 5000 \qquad Var\left(\sum_1^{100} X_i\right) = 1250000$$

$$P\left(\sum_1^{100} X_i > 8000\right) = P\left(N(0,1) \geq \frac{8000 - 5000}{\sqrt{1250000}}\right)$$

$$= P(N(0,1) \geq 2.68) = 0.0037$$

6. $S \sim Bin\left(10000, \frac{1}{6}\right)$ $\quad E[S] = np = \frac{500}{3}$

$$Var(S) = \frac{2500}{18} =$$

$$P(150 \leq X < 200) = P(149.5 \leq X \leq 199.5)$$

$$= P\left(\frac{149.5 - \frac{500}{3}}{\sqrt{\frac{2500}{18}}} \leq N(0,1) \leq \frac{199.5 - \frac{500}{3}}{\sqrt{\frac{2500}{18}}}\right) = P(-1.46 \leq N(0,1) \leq 2.78)$$

$$= 1 - 0.0027 - 0.0722 = 0.9251$$

7. $E[Y] = \frac{1}{2}$ $Var(Y) = \frac{1}{2^2}$

$E[\sum_1^{200} Y_i] = 100$ $Var(\sum_1^{200} Y_i) = 50$

$Pr(\sum_1^{200} Y_i \geq 110) \simeq P(N(0,1) \geq \frac{110-100}{\sqrt{50}})$

$= P(N(0,1) \geq 1.41) = 0.0793$

8. $S \sim Bin(n=50, p=0.8)$

$E[S] = np = 40$ $Var(S) = 8$

$P(S \leq 30) = P(N(0,1) \leq \frac{30.5-40}{\sqrt{8}}) = P(N(0,1) \leq -3.36)$

$= 0.000233$

9. $E[X] = 150$ $Var(X) = 55^2$

$E[\sum_1^{30} X_i] = (30)(150) = 4500$ $Var(\sum_1^{30} X_i) = (30)(55)^2 =$

$P(\sum_1^{30} X_i > 5000) = P(N(0,1) \geq \frac{5000-450}{55\sqrt{30}}) = P(N(0,1) \geq 1.66)$

$= 0.0485$

10.

$$0.025 = P(\bar{x} > 100.1) = P\left(N(0,1) \geq \frac{100.1 - 100}{\sqrt{\frac{4}{n}}}\right)$$

$$\sqrt{n}\,(0.05) = 1.96 \qquad n = \left(\frac{1.96}{0.05}\right)^2 = 1536.64$$