Homework 3

1) $\hat{\theta}_1 = a\bar{Y}$

$$E[Y] = \int_0^\theta y \frac{3y^2}{\theta^3}\,dy = \frac{3y^4}{4\theta^3}\Big]_0^\theta = \frac{3\theta}{4}$$

$$E[Y^2] = \int_0^\theta y^2 \frac{3y^2}{\theta^3}\,dy = \frac{3}{5}\theta^2$$

$$\theta = E[\hat{\theta}_1] = a\frac{3}{4}\theta \qquad a = \frac{4}{3}$$

$$MSE(\hat{\theta}_1) = Var\left(\frac{4}{3}\bar{Y}\right) = \frac{16}{9}\frac{Var(Y)}{n} = \frac{16}{9n}\left(\frac{3\theta^2}{5} - \left(\frac{3\theta}{4}\right)^2\right)$$

$$= \frac{16\theta^2}{9n}\left(\frac{3}{5} - \frac{9}{16}\right) = \frac{16\theta^2}{9n}\frac{3}{80} = \frac{\theta^2}{15n}$$

$\hat{\theta}_2 = bY_{(n)}$ $\qquad f_{Y_{(n)}}(y) = n(F(y))^{n-1}f(y) = n\left(\frac{y^3}{\theta^3}\right)^{n-1}\frac{3y^2}{\theta^3} = \frac{3n\,y^{3n-1}}{\theta^{3n}}$

$$E[Y_{(n)}] = \int_0^\theta y\frac{3n\,y^{3n-1}}{\theta^{3n}} = \frac{3n}{3n+1}\theta \qquad E[Y_{(n)}^2] = \frac{3n\,\theta^2}{3n+2}$$

$$\theta = E[\hat{\theta}_2] = b\frac{3n}{3n+1}\theta \qquad b = \frac{3n+1}{3n}$$

$$MSE(\hat{\theta}) = MSE\left(\frac{3n+1}{3n}Y_{(n)}\right) = \frac{(3n+1)^2}{(3n)^2}\left(\frac{3n\theta^2}{3n+2} - \left(\frac{3n\theta}{3n+1}\right)^2\right)$$

$$= \frac{(3n+1)^2}{(3n)^2}\theta^2\left(\frac{3n(3n+1)^2 - 9n^2(3n+2)}{(3n+2)(3n+1)^2}\right) = \frac{\theta^2}{(3n)(3n+2)}(9n^2+6n+1-9n^2-6n)$$

$$= \frac{\theta^2}{(3n+2)(3n)}$$

$(3n+2)3n \geq 15n$ $\qquad 9n^2-6n+2 \geq 0$ $\qquad \frac{3 \pm \sqrt{9-72}}{18}$

$\hat{\theta}_2$ is preferred

2. $\hat{\theta}_1 = a\bar{Y}$

$$E[Y] = \int_0^\infty \frac{m\, y^m e^{-\frac{y^m}{\theta}}}{\theta} dy = \int_0^\infty \frac{m}{\theta}\, \theta x\, e^{-x}\, \theta^{\frac{1}{m}} \frac{1}{m} x^{\frac{1}{m}-1} dx$$

$$\frac{y^m}{\theta} = x \quad y = \theta^{\frac{1}{m}} x^{\frac{1}{m}}$$

$$= \int_0^\infty \theta^{\frac{1}{m}} x^{\frac{1}{m}} e^{-x} dx = \theta^{\frac{1}{m}} \Gamma\left(\frac{m+1}{m}\right)$$

$$E[Y^2] = \int_0^\infty y^2 \frac{m y^{m-1} e^{-\frac{y^m}{\theta}}}{\theta} dy = \int_0^\infty \theta^{\frac{2}{m}} x^{\frac{2}{m}} e^{-x} dx = \theta^{\frac{2}{m}} \Gamma\left(\frac{m+2}{m}\right)$$

$$E[Y^m] = \int_0^\infty y^m \frac{m y^{m-1} e^{-\frac{y^m}{\theta}}}{\theta} = \int_0^\infty \theta x\, e^{-x} dx = \theta$$

$$\frac{y^m}{\theta} = x$$

$$E[Y^{2m}] = \int_0^\infty y^{2m} \frac{m y^{m-1}}{\theta} e^{-\frac{y^m}{\theta}} dy = \int_0^\infty \theta^2 x^2 e^{-x} dx = 2\theta^2$$

$$Var(Y^m) = \theta^2$$

$\hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^{n} Y_i^m$ is an unbiased estimator of θ

$$MSE(\hat{\theta}_2) = \frac{\sigma^2}{n} = \frac{\theta^2}{n}$$

$$MSE(\bar{Y}) = \theta^{\frac{2}{m}}\left(\Gamma\left(\frac{m+2}{m}\right) - \theta^{\frac{2}{m}}\left(\Gamma\left(\frac{m+1}{m}\right)\right)^2\right) + \left(\theta^{\frac{1}{m}}\Gamma\left(\frac{m+1}{m}\right) - \theta\right)^2$$

3. $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$

$$E[s] = E\left[\frac{\sigma}{\sqrt{n-1}} \sqrt{\chi^2(n-1)}\right] = \frac{\sigma}{\sqrt{n-1}} \int_0^\infty x^{\frac{1}{2}} \frac{x^{\frac{n-1}{2}-1} e^{-\frac{x}{2}}}{\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} \, dx$$

$$= \frac{\Gamma\left(\frac{n}{2}\right) 2^{n/2}}{\sqrt{n-1}\;\Gamma\left(\frac{n-1}{2}\right) 2^{\frac{n-1}{2}}} = \frac{\Gamma\left(\frac{n}{2}\right)\sqrt{2}}{\sqrt{n-1}\;\Gamma\left(\frac{n-1}{2}\right)}$$

$\frac{\Gamma\left(\frac{n-1}{2}\right)\sqrt{n-1}}{\Gamma\left(\frac{n}{2}\right)\sqrt{2}} s$ is an unbiased estimator of σ

4.

$$\sum_{i=1}^{n} y_i = 441.6 \qquad \sum_{i=1}^{n} y_i^2 = 6555.64$$

$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{441.6}{30} = 14.72$ is a point estimate of μ

$$\sum (y_i-\bar{y})^2 = \sum y_i^2 - n(\bar{y})^2 = 6555.64 - (30)(14.72)^2 = 55.288$$

$s^2 = \frac{1}{n-1}\sum (y_i-\bar{y})^2 = \frac{55.288}{29} = 1.90648$ is a point estimate of σ^2

A 90% confidence interval for μ is

$$\bar{y} \pm t_{0.05}\frac{s}{\sqrt{n}} = 14.72 \pm (1.6991)\frac{\sqrt{1.90648}}{\sqrt{30}} = 14.72 \pm 0.4285$$

$$= [14.2915, 15.1485]$$

A 95% confidence interval for σ^2 is

$$\left[\frac{(n-1)s^2}{\chi^2_{\frac{\alpha}{2}}}, \frac{(n-1)s^2}{\chi^2_{1-\frac{\alpha}{2}}}\right] = \left[\frac{(29)(1.90648)}{45.7222}, \frac{(29)(1.90648)}{16.0471}\right]$$

$$= [1.2092, 3.445]$$

5.

a) $0.5 = z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} = 2.575 \frac{1.6}{\sqrt{n}}$

$n = \left(\frac{(2.575)(1.6)}{0.5}\right)^2 = 67.8976$

b) $0.5 = z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} = (1.96) \frac{(1.6)}{\sqrt{n}}$

$n = \left(\frac{(1.96)(1.6)}{0.5}\right)^2 = 39.3380$

6. $\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{1}{n}\hat{p}(1-\hat{p})}$

$\hat{p} = \frac{190}{250} = 0.76 \qquad z_{\frac{\alpha}{2}} = 1.96$

$0.76 \pm 1.96 \sqrt{\frac{1}{250}(0.76)(0.24)} = 0.76 \pm 0.0529$

$= (0.7071, 0.8129)$

7. Y_1 = no. of male voters prefering the candidate

$Y_1 \sim Bino(n_1, p_1)$

Y_2 = no. of female voters prefering the candidate

$Y_2 \sim Binom(n_2, p_2)$

$$\frac{Y_1}{n_1} - \frac{Y_2}{n_2} = \frac{132}{200} - \frac{90}{154} = 0.0940$$

$$Var\left(\frac{Y_1}{n_1} - \frac{Y_2}{n_2}\right) = \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}$$

$$\hat{\sigma}^2_{\hat{p}_1-\hat{p}_2} = \frac{1}{n_1}\frac{Y_1}{n_1}\left(1-\frac{Y_1}{n_1}\right) + \frac{1}{n_2}\frac{Y_2}{n_2}\left(1-\frac{Y_2}{n_2}\right)$$

$$= \frac{1}{200}\frac{132}{200}\left(1-\frac{132}{200}\right) + \frac{1}{154}\frac{90}{154}\left(1-\frac{90}{154}\right) = 0.0026669$$

$z_{0.005} = 2.575.$

The 99% confidence interval for $p_1 - p_2$ is

$$\frac{Y_1}{n_1} - \frac{Y_2}{n_2} \pm z_{\frac{\alpha}{2}} \hat{\sigma}_{\hat{p}_1-\hat{p}_2} = 0.0940 \pm (2.575)(0.05164)$$

$$= 0.0940 \pm 0.1330 = (-0.039, 0.227)$$

8. The confidence interval for p, given that $Y \sim Binom(n,p)$

is $\frac{y}{n} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{1}{n}\frac{y}{n}\left(1-\frac{y}{n}\right)}$

$z_{0.05} = 1.645$, We do not know $\frac{y}{n}$, so, we take $\frac{y}{n} = \frac{1}{2}$

$1.645\sqrt{\frac{1}{n}\frac{1}{2}\frac{1}{2}} = 0.02$ $\quad n = \left(\frac{1.645}{2(0.02)}\right)^2 = 1691.267$

9.

$\bar{y}_1 = 3.1 \quad n_1 = 10, \quad s_1 = 0.5$

$\bar{y}_2 = 2.7 \quad n_2 = 8 \quad s_2 = 0.7$

the $1-\alpha$ confidence interval of $\mu_1 - \mu_2$ is $\bar{y}_1 - \bar{y}_2 \pm t_{\frac{\alpha}{2}} \sqrt{S_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$

where $S_p^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2} = \frac{9(0.5)^2 + 7(0.7)^2}{10+8-2} = 0.355$

$t_{0.025} = 2.12$

The 95% confidence interval for $\mu_1 - \mu_2$ is

$3.1 - 2.7 \pm 2.12\sqrt{0.355}\sqrt{\frac{1}{10}+\frac{1}{8}} = 0.4 \pm 0.5992$

$= (-0.1992, \ 0.9992)$

10. $n=9$ $\sum x_i = 148$ $\sum x_i^2 = 2792$, $\bar{x} = 16.441$

$\sum (x_i-\bar{x})^2 = \sum x_i^2 - n(\bar{x})^2 = 2792 - 9(1.644)^2 = 359.5376$

$s_1^2 = \frac{\sum (x_i-\bar{x})^2}{n-1} = \frac{359.5376}{8} = 44.9422$

$m=10$ $\sum y_i = 233$ $\sum y_i^2 = 6263$ $\bar{y} = 23.3$

$\sum (y_i-\bar{y})^2 = \sum y_i^2 - m(\bar{y})^2 = 6263 - 10(23.3)^2 = 834.1$

$s_2^2 = \frac{\sum (y_i-\bar{y})^2}{m-1} = \frac{834.1}{9} = 92.6777$

$\frac{s_1^2}{s_2^2} = \frac{44.9422}{92.6777} = 0.4893$

$\frac{\frac{s_1^2}{\sigma_2^2}}{\frac{s_2^2}{\sigma_1^2}} \sim F(8,9)$

We find a and b such that $P(a \le F(8,9) \le b) = 0.95$

$0.025 = P(F(8,9) \ge 4.10)$

$0.025 = P(F(9,8) \ge 4.36) = P(\frac{1}{4.36} \ge F(8,9))$

$0.95 = P(\frac{1}{4.36} \le F(8,9) \le 4.10) = P(\frac{1}{4.36} \le \frac{\frac{s_1^2}{\sigma_1^2}}{\frac{s_2^2}{\sigma_2}} \le 4.10)$

$P(\frac{0.4893}{4.10} \le \frac{\sigma_1^2}{\sigma_2^2} \le (4.36)(0.4893)) = P(0.1182 \le \frac{\sigma_1^2}{\sigma_2^2} \le 2.1143)$

The confidence interval for $\frac{\sigma_1^2}{\sigma_2}$ is

$(0.1182, 2.1143)$