1.

$$f(y|\theta,\alpha) = \frac{\alpha\,\theta^{\alpha}}{y^{\alpha+1}}\, I(y>\theta)$$

1

$$L(y_1 \ldots y_n | \theta, \alpha) = \prod_{j=1}^{n} \frac{\alpha\,\theta^{\alpha}}{y_j^{\alpha+1}}\, I(y_j > \theta) = \underbrace{\frac{\alpha^{n}\,\theta^{n\alpha}}{\left(\prod_{j=1}^{n} y_j\right)^{\alpha+1}}\, I(y_{(1)} > \theta)}_{g\left(y_{(1)},\ \prod_{j=1}^{n} y_j\right)}\ \underbrace{1}_{h(y_1 - y_n)}$$

$Y_{(1)}$ and $\prod_{j=1}^{n} Y_j$ are jointly sufficent for θ and α

2

$$f(y|\theta) = \frac{3y^5 e^{-y^3/\theta}}{\theta^2}, \text{ if } y>0$$

$$L(y_1 \ldots y_n|\theta) = \prod_{j=1}^{n} f(y_j,\theta) = \prod_{j=1}^{n} \frac{3y_j^5 e^{-y_j^3/\theta}}{\theta^2} = \frac{3^n \prod_{j=1}^{n} y_j^5 e^{-\sum_{j=1}^{n} y_j^3/\theta}}{\theta^{2n}}$$

$$= \underbrace{\frac{3^n e^{-\sum_{j=1}^{n} y_j^3/\theta}}{\theta^{2n}}}_{g(U,\theta)} \underbrace{\prod_{j=1}^{n} y_j^5}_{h(y_1 - y_n)}$$

$\underline{U = \sum_{j=1}^{n} y_j^3}$ is a sufficient statistic for θ

$$E[U] = nE[Y^3] = n\int_0^\infty y^3 \frac{3y^5 e^{-y^3/\theta}}{\theta^2} dy$$

$$\frac{y^3}{\theta} = x \quad 3y^2 dy = dx$$

$$= n\int_0^\infty \theta x \cdot x \; e^{-x} dx = 2n\theta$$

$$\boxed{\frac{1}{2n}\sum_{j=1}^{n} y_j^3 \text{ is a MVUE of } \theta}$$

$$E[Y^6] = \int_0^\infty y^6 \frac{3y^5 e^{-y^3/\theta}}{\theta^2} dy = \int_0^\infty \theta^2 x^2 \, x e^{-x} dx = 6\theta^2$$

$$\frac{y^3}{\theta} = x$$

$$\mathrm{Var}(Y^3) = 2\theta^2$$

$$E\left[\left(\sum_{j=1}^{n} Y_1^3\right)^2\right] = Var\left(\sum_{j=1}^{n} Y_1^3\right) + \left(E\left[\sum_{j=1}^{n} Y_1^3\right]\right)^2$$

$$= n\, 2\theta^2 + (n\, 2\theta)^2 = 2n\theta^2 + 4n^2\theta^2 = 2n(2n+1)\,\theta^2$$

$$\boxed{\frac{\left(\sum_{j=1}^{n} Y_1^3\right)^2}{2n(2n+1)} \text{ is a MVUE of } \theta^2.}$$

3

$$f(y|\theta) = \frac{\alpha y^{\alpha-1}}{\theta^\alpha} \quad \text{if } 0<y<\theta$$

$Y_{(n)}$ is a sufficient statistic for θ

$$L(y_1 \ldots y_n|\theta) = \prod_{j=1}^{n} \frac{\alpha y_j^{\alpha-1}}{\theta^\alpha} I(0<y_j<\theta) = \frac{\alpha^n (\prod y_i)^{\alpha-1}}{\theta^{n\alpha}} I(0<Y_{(1)}, Y_{(n)}<\theta)$$

$$= \underbrace{\frac{I(Y_{(n)}<\theta)}{\theta^{n\alpha}}}_{g(u,\theta)} \quad \underbrace{\alpha^n (\prod y_i)^{\alpha-1} I(0<Y_{(1)})}_{h(y_1 \ldots y_n)}$$ $Y_{(n)}$ is a sufficient of θ

$\boxed{Y_{(n)} \text{ is a sufficient statistic of } \theta}$

$$F(y) = \left(\frac{y}{\theta}\right)^\alpha$$

$$f_{Y_{(n)}}(y) = n\,(F(y))^{n-1} f(y) = n\left(\left(\frac{y}{\theta}\right)^\alpha\right)^{n-1} \frac{\alpha y^{\alpha-1}}{\theta^\alpha} = \frac{n\alpha\, y^{n\alpha-1}}{\theta^{n\alpha}}, \quad \text{if } 0<y<\theta$$

$$E[Y_{(n)}] = \int_0^\theta y\, \frac{n\alpha\, y^{n\alpha-1}}{\theta^{n\alpha+1}}\, dy = \frac{n\alpha}{(n\alpha+1)}\theta$$

$\boxed{\frac{(n\alpha+1)}{n\alpha} Y_{(n)} \text{ is the UMVUE of } \theta}$

$$E[Y^2] = \frac{\alpha}{\alpha+2}\theta^2 \qquad Var(Y) = \frac{\alpha}{\alpha+2}\theta^2 - \left(\frac{\alpha}{\alpha+1}\theta\right)^2 = \frac{\alpha\theta^2}{(\alpha+2)(\alpha+1)^2}$$

$$= \frac{\alpha\theta^2(\alpha+1)^2 - \alpha^2(\alpha+2)\theta^2}{(\alpha+2)(\alpha+1)^2} = \frac{\alpha\theta^2}{(\alpha+2)(\alpha+1)^2}$$

$$E[Y_{(n)}^2] = \frac{n\alpha\,\theta^2}{(n\alpha+2)}$$

$\boxed{\frac{(n\alpha+2)}{n\alpha} \frac{\alpha}{(\alpha+2)(\alpha+1)^2} Y_{(n)}^2 \text{ is the MVUE of } Var(Y)}$

4. $f(y,\theta) = e^{-(y-\theta)}\, I(y>\theta)$

$Y_{(1)}$ is a sufficient statistic of θ.

$F(y) = 1 - e^{-(y-\theta)}$

$f_{Y_{(1)}}(y) = n\,(1-F(y))^{n-1} f(y) = n\left(e^{-(y-\theta)}\right)^{n-1} e^{-(y-\theta)} = n e^{-n(y-\theta)}$, if $y>\theta$

$$E[Y_{(1)}] = \int_\theta^\infty y\, n e^{-n(y-\theta)}\, dy = \int_0^\infty \left(\theta + \frac{x}{n}\right) e^{-x}\, dx = \theta + \frac{1}{n}$$

$n(y-\theta) = x$

$dy = \frac{dx}{n}$

$Y_{(1)} - \frac{1}{n}$ is a MVUE of θ

$$E[Y_{(1)}^2] = \int_\theta^\infty y^2\, n e^{-n(y-\theta)}\, dy = \int_\theta^\infty \left(\theta + \frac{x}{n}\right)^2 e^{-x}\, dx = \theta^2 + \frac{2\theta}{n} + \frac{2}{n^2}$$

$$E\left[Y_{(1)}^2 - \frac{2}{n} Y_{(1)}\right] = \theta^2 + \frac{2\theta}{n} + \frac{2}{n^2} - \frac{2}{n}\left(\theta + \frac{1}{n}\right) = \theta^2$$

$Y_{(1)}^2 - \frac{2}{n} Y_{(1)}$ is the MVUE of θ^2

5 $p(y|\theta) = \theta y^{\theta-1}$ if $0 < y < 1$

$$E[Y] = \int_0^1 y\,\theta y^{\theta-1}\,dy = \frac{\theta}{\theta+1}$$

$$\bar{Y} = \frac{\theta}{\theta+1} \qquad \hat{\theta} = \frac{\bar{Y}}{1-\bar{Y}} \text{ is the MME of } \theta$$

$$\hat{\theta} = \frac{\bar{Y}}{1-\bar{Y}} \xrightarrow{P} \frac{\frac{\theta}{\theta+1}}{1-\frac{\theta}{\theta+1}} = \theta$$

$$L(\theta) = \theta^n \left(\prod Y_i\right)^{\theta-1}$$

$$\ln L(\theta) = n \ln\theta + (\theta-1)\sum_{i=1}^n \ln Y_i$$

$$\frac{\partial}{\partial\theta}\ln L(\theta) = \frac{n}{\theta} + \sum_{i=1}^n \ln Y_i = 0, \quad \theta = \frac{-n}{\sum_{i=1}^n \ln Y_i}$$

The MLE of θ is $\hat{\theta} = \dfrac{-n}{\sum_{i=1}^n \ln Y_i}$

$$E[\ln Y] = \int_0^1 \ln y\,\theta y^{\theta-1}\,dy = \int_0^\infty (-x)\,\theta\, e^{-x(\theta-1)} e^{-x}\,dx = \int_0^\infty -x\;\theta e^{-\theta x}\,dx$$

$y = e^{-x} \quad dy = -e^{-x}dx$ $\qquad \theta x = y$

$$= -\int_0^\infty \frac{y e^{-y}}{\theta}\,dy = \frac{-1}{\theta}$$

$$\hat{\theta}_n = \frac{-n}{\sum_{i=1}^n \ln Y_i} \xrightarrow{P} \frac{(-1)}{(-1/\theta)} = \theta.$$

6. $f(y|\theta) = \dfrac{2y^3 e^{-\frac{y^2}{\theta^2}}}{\theta^4}$, if $0 < y$

(a) $E[Y] = \displaystyle\int_0^\infty y \frac{2y^3 e^{-\frac{y^2}{\theta^2}}}{\theta^4} dy = \int_0^\infty y \frac{y^2}{\theta^2} e^{-\frac{y^2}{\theta^2}} \frac{2y}{\theta^2} dy$

$\left(\frac{y}{\theta}\right)^2 = u$

$2\frac{y}{\theta^2} dy = du$

$= \displaystyle\int_0^\infty \theta \sqrt{u}\, u\, e^{-u} du = \int_0^\infty \theta u^{3/2} e^{-u} du = \theta \frac{3}{4}\sqrt{\pi}$

$\bar{Y} = \frac{3\sqrt{\pi}}{4}\theta \qquad \hat{\theta} = \frac{4}{3\sqrt{\pi}}\bar{Y}$

$\hat{\theta} \xrightarrow{P} \frac{4}{3\sqrt{\pi}} E[Y] = \theta$

(b) $L(\theta) = \displaystyle\prod_{j=1}^n \frac{2y_j^3 e^{-\frac{y_j^2}{\theta^2}}}{\theta^4} = \frac{2^n (\prod y_j)^3 e^{-\sum y_j^2/\theta^2}}{\theta^{4n}}$

$\ln L(\theta) = \ln(2^n(\prod y_j^3)) - \frac{\sum y_j^2}{\theta^2} - 4n \ln\theta$

$\frac{\partial}{\partial\theta} \ln L(\theta) = \frac{2\sum y_j^2}{\theta^3} - \frac{4n}{\theta} \qquad \hat{\theta} = \left(\frac{1}{2n}\sum_{j=1}^n y_j^2\right)^{1/2}$

$E[Y_j^2] = \displaystyle\int_0^\infty y^2 \frac{2y^3 e^{-y^2/\theta^2}}{\theta^4} dy = \int_0^\infty \theta^2 u\, u\, e^{-u} du = 2\theta^2$

$\hat{\theta} \xrightarrow{P} \left(\frac{2\theta^2}{2}\right)^{1/2} = \theta$

$\left(\frac{E[Y^2]}{2}\right)^{1/2}$

7. $E[Y] = \int_0^\theta y \frac{\alpha y^{\alpha-1}}{\theta^\alpha} dy = \frac{\alpha}{\alpha+1}\theta$

$\bar{Y} = \frac{\alpha}{\alpha+1}\theta \quad \hat{\theta} = \frac{(\alpha+1)\bar{Y}}{\alpha}$ is the MME of θ

$\hat{\theta} \xrightarrow{P} \frac{\alpha+1}{\alpha}\frac{\alpha}{\alpha+1}\theta = \theta$

$E[Y^2] = \int_0^\theta y^2 \frac{\alpha y^{\alpha-1}}{\theta^\alpha} dy = \frac{\alpha}{\alpha+2}\theta^2$

$Var(Y) = \frac{\alpha}{\alpha+2}\theta^2 - \left(\frac{\alpha}{\alpha+1}\theta\right)^2 = \frac{\alpha\theta^2}{(\alpha+1)^2(\alpha+2)}$

$MSE\left(\frac{(\alpha+1)}{\alpha}\bar{Y}\right) = \frac{(\alpha+1)^2}{n\alpha^2}Var(Y) = \frac{\theta^2}{n\alpha(\alpha+2)}$

The MLE of θ is $Y_{(n)}$

$f_{Y_{(n)}}(y) = n\left(\frac{y^\alpha}{\theta^\alpha}\right)^{n-1}\frac{\alpha y^{\alpha-1}}{\theta^\alpha} = \frac{n\alpha y^{n\alpha-1}}{\theta^{n\alpha}}$

$E[Y_{(n)}] = \frac{n\alpha}{n\alpha+1}\theta \to \theta$, $E[Y_{(n)}^2] = \frac{n\alpha}{n\alpha+2}\theta^2 \to \theta^2$

So, $Y_{(n)}$ is consistent of θ. It is not unbiased

$Var(Y_{(n)}) = \frac{n\alpha}{n\alpha+2}\theta^2 - \left(\frac{n\alpha\theta}{n\alpha+1}\right)^2 = \frac{n\alpha\theta^2}{(n\alpha+1)^2(n\alpha+2)}$

$MSE(Y_{(n)}) = \frac{n\alpha\theta^2}{(n\alpha+1)^2(n\alpha+2)} + \left(\frac{n\alpha\theta}{n\alpha+1} - \theta\right)^2 = \frac{n\alpha\theta^2}{(n\alpha+1)^2(n\alpha+2)} + \frac{\theta^2}{(n\alpha+1)^2}$

$= \frac{2\theta^2}{(n\alpha+1)(n\alpha+2)}$

8.

$$E[Y] = \int_\theta^\infty y\, e^{-(y-\theta)}\, dy = \int_0^\infty (\theta + x)\, e^{-x}\, dx = \theta + 1$$

$\bar{Y} = \theta + 1$ $\quad \hat{\theta}_1 = \bar{Y} - 1$ is the MME of θ

$\bar{Y} - 1 \xrightarrow{P} E[Y] - 1 = \theta$, $\hat{\theta}_1$ is consistent of θ

$$Var(\hat{\theta}_1) = \frac{\sigma^2}{n} = \frac{1}{n}$$

$$E[Y^2] = \int_0^\infty y^2\, e^{-(y-\theta)}\, dy = \int_0^\infty (\theta + x)^2\, e^{-x}\, dx = \theta^2 + 2\theta + 2$$

$y - \theta = x$

$$Var(Y) = \theta^2 + 2\theta + 2 - (\theta + 1)^2 = 1$$

The MLE is $Y_{(1)}$

$$f_{Y_{(1)}}(y) = n(1 - F(y))^{n-1} f(y) = n\, e^{-n(y-\theta)}$$

$$E[Y_{(1)}] = \int_0^\infty y\, n\, e^{-n(y-\theta)}\, dy = \int_0^\infty \left(\theta + \frac{x}{n}\right) e^{-x}\, dx = \theta + \frac{1}{n}$$

$$E[Y_{(1)}^2] = \int_0^\infty y^2\, n\, e^{-n(y-\theta)}\, dy = \int_0^\infty \left(\theta + \frac{x}{n}\right)^2 e^{-x}\, dx = \theta^2 + \frac{2}{n} + \frac{2}{n^2}$$

$$Var(Y_{(1)}) = \frac{1}{n^2}$$

$$MSE(Y_{(1)}) = Var(Y_{(1)}) + (E[Y_{(1)}] - \theta)^2 = \frac{2}{n^2}$$

For $n = 1$ $\quad MSE(\bar{Y} - 1) < MSE(Y_{(1)})$

For $n \geq 3$ $\quad MSE(\bar{Y} - 1) > MSE(Y_{(1)})$

a.

$$L(\theta) = \prod_{j=1}^{n} \frac{m y_j^{m-1} e^{-\frac{y_j^m}{\theta}}}{\theta} = \frac{m^n (\prod y_j)^{m-1} e^{-\frac{\sum y_j^m}{\theta}}}{\theta^n}$$

$$\ln L(\theta) = \ln(m^n) + \ln((\prod y_j)^{m-1}) - \frac{\sum y_j^m}{\theta} - n \ln \theta$$

$$\frac{\partial}{\partial \theta} \ln L(\theta) = \frac{\sum_{j=1}^{n} y_j^m}{\theta^2} - \frac{n}{\theta} = 0 \qquad \hat{\theta} = \frac{1}{n} \sum_{j=1}^{m} y_j^m$$

$$\ln f(y|\theta) = \ln(m y^{m-1}) - \frac{y^m}{\theta} - \ln \theta$$

$$\frac{\partial}{\partial \theta} \ln f(y|\theta) = \frac{y^m}{\theta^2} - \frac{1}{\theta}$$

$$\frac{\partial^2}{\partial \theta^2} \ln f(y|\theta) = \frac{-2 y^m}{\theta^3} + \frac{1}{\theta^2}$$

$$-E\left[\frac{\partial^2}{\partial \theta^2} \ln f(y|\theta)\right] = \frac{2}{\theta^3} E[y^m] - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

$$E[y^m] = \int_0^\infty y^m \frac{m y^{m-1}}{\theta} e^{-\frac{y^m}{\theta}} dy = \int_0^\infty x \theta e^{-x} dx = \theta$$

$$\frac{y^m}{\theta} = x$$

$$\hat{\theta} \pm z_{\frac{\alpha}{2}} \frac{1}{\sqrt{n E\left[-\frac{\partial^2}{\partial \theta^2} \ln f(Z,\theta)\right]_{\theta=\hat{\theta}}}} = \hat{\theta} \pm z_{\frac{\alpha}{2}} \frac{\hat{\theta}}{\sqrt{n}}$$

10 $f(y|\lambda) = e^{-\lambda} \dfrac{\lambda^y}{y!}$, $y = 0, 1 - -$

$$L(y) = \prod_{j=1}^{n} e^{-\lambda} \frac{\lambda^{y_j}}{y_j!} = e^{-\lambda n} \frac{\lambda^{n\bar{y}}}{\prod y_i!}$$

$$\ln L(\lambda) = -\lambda n + n\bar{y} \ln \lambda - \ln \prod y_i!$$

$$\frac{\partial}{\partial \lambda} \ln L(\lambda) = -n + \frac{n\bar{y}}{\lambda} \qquad \lambda = \bar{y}$$

$\hat{\lambda} = \bar{y}$ is the MLE of λ

$$\frac{\partial}{\partial \lambda} e^{-\lambda} = -e^{-\lambda}$$

$$\ln f(y, \lambda) = -\lambda + y \ln \lambda - \ln y!$$

$$\frac{\partial}{\partial \lambda} \ln f(y, \lambda) = -1 + \frac{y}{\lambda} \qquad \frac{\partial^2}{\partial y^2} \ln f(y, \lambda) = -\frac{y}{\lambda^2}$$

$$-E\left[\frac{\partial^2}{\partial \lambda^2} \ln f(y, \lambda)\right] = \frac{1}{\lambda}$$

$$t(\lambda) \pm z_{\frac{\alpha}{2}} \frac{|t'(\lambda)|}{\sqrt{n E[-\frac{\partial^2}{\partial \lambda^2} \ln f(y, \lambda)]\Big|_{\lambda = \hat{\lambda}}}} = e^{-\hat{\lambda}} \pm z_{\frac{\alpha}{2}} \frac{e^{-\hat{\lambda}}}{\sqrt{n \frac{1}{\hat{\lambda}}}}$$

$$= e^{-\bar{y}} \pm z_{\frac{\alpha}{2}} \frac{\sqrt{\bar{y}}\, e^{-\bar{y}}}{\sqrt{n}}$$