Math–448. Extra problems.

Neyman–Pearson Lemma

1. Let $X_1, \ldots, X_n$ be random sample of size n from the density

$$f(y|\theta) = \begin{cases} \frac{10y^9 e^{-y^{10}/\theta}}{\theta} & \text{if } 0 < y, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is unknown. Find the uniformly most powerful test for $H_0 : \theta = \theta_0$, versus $H_a : \theta < \theta_0$ (we just to write the rejection region in the form $\{(y_1, \ldots, y_n) : T(y_1, \ldots, y_n) \geq k'\}$, where $T(y_1, \ldots, y_n)$ is a statistic which you need to find).

2. Let $X_1, \ldots, X_n$ be random sample of size $n = 20$ from the density $N(0, \theta)$, $\theta > 0$. Find the uniformly most powerful level 95% test for $H_0 : \theta = 4$, versus $H_1 : \theta < 4$ (determine the region completely).

3. Let $X_1, \ldots, X_n$ be a random sample from $f(x|\theta) = \frac{1}{\theta} m x^{m-1} e^{-x^m/\theta}$, $x > 0$, where $m > 0$ is known and $\theta > 0$ is unknown. Find the uniformly most powerful level α test for the problem $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$.

4. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with parameter $\theta > 0$. Find the uniformly most powerful level α test for $H_0 : \theta = 1$, versus $H_1 : \theta > 1$ (we just to write the rejection region in the form $\{(x_1, \ldots, x_n) : T(x_1, \ldots, x_n) \geq c\}$, where $T(x_1, \ldots, x_n)$ is a statistic which you need to find).

5. Let $X_1, \ldots, X_n$ be random sample of size n from a normal density with mean zero and variance σ^2. Find the uniformly most powerful test for $H_0 : \sigma^2 = \sigma_0^2$, versus $H_a : \sigma^2 < \sigma_0^2$, where $\sigma_0^2 > 0$.

6. Let $X_1, \ldots, X_n$ be random sample of size n from the density $f(x|\theta) = \theta x^{\theta-1}$, $0 < x < 1$, $\theta > 0$. Find the uniformly most powerful level α test for $H_0 : \theta \leq \theta_0$, versus $H_1 : \theta > \theta_0$.

7. Let $X_1, \ldots, X_n$ be a random sample from $f(x) = \frac{x^{\theta-1} e^{-x}}{\Gamma(\theta)}$, $x > 0$; and zero otherwise; $\theta > 0$. Use the Neyman–Pearson Lemma to find the uniformly most powerful level α test for testing $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$.

Likelihood ratio test

8. Let $X_1, \ldots, X_n$ be random sample of size n from a pdf from the density $f(x, \theta) = \theta^x (1-\theta)^{1-x}$, $x = 0, 1$, $0 \leq \theta \leq 1$. Show that the likelihood ratio test for $H_0 : \theta = \frac{1}{2}$, versus $H_1 : \theta \neq \frac{1}{2}$ reject H_0 if $|\bar{x} - \frac{1}{2}| \geq c$, where c is a constant. Hint: first show that the LRT reject H_0 if $\bar{x}^{\bar{x}}(1-\bar{x})^{1-\bar{x}} \geq c$, where c is a constant. Observe that the function $f(x) = \ln(x^x(1-x)^{1-x})$, $0 \leq x \leq 1$, satifies $f(x) = f(1-x)$ and it has a minimum at $x = \frac{1}{2}$.

9. A random sample of size n is to be used to test the null hypothesis that the parameter θ of an exponential propulation equals θ_0, against the alternative that it does not equal θ_0. Find an expression for the likelihood ratio statistic. Use the previous result to show that the critical region of the likelihood ratio test can be written as $\frac{\bar{x}}{\theta_0} e^{-\frac{\bar{x}}{\theta_0}} \leq c$. Prove that the equation as $\frac{\bar{x}}{\theta_0} e^{-\frac{\bar{x}}{\theta_0}} \leq c$ is equivalent to $c_1 \leq \frac{\bar{x}}{\theta_0} \leq c_2$, where $c_1 < 1 < c_2$. Hint: The function $f(x) = xe^{-x}$ has a maximum at $x = 1$.

10. Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$, where $\mu \in I\!R$ and $\sigma^2 > 0$ are unknown parameters. Find the rejection region for the likelihood ratio test $H_0 : \mu = \mu_0$, versus $H_1 : \mu \neq \mu_0$.

11. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with parameter $\theta > 0$. Find the likelihood ratio test of $H_0 : \theta = \theta_0$ against $H_a : \theta \neq \theta_0$, where $\theta_0 > 0$ (we just to write the rejection region in the form $\{(y_1, \ldots, y_n) : T(y_1, \ldots, y_n) \geq k'\}$, where $T(y_1, \ldots, y_n)$ is a statistic which you need to find).

12. Let $X_1, \ldots, X_n$ be a random sample from the density

$$f(y|\theta) = \begin{cases} \frac{4y^3 e^{-\frac{y^4}{\theta}}}{\theta} & \text{if } 0 < y, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is unknown parameter. Find the maximum likelihood estimator of θ. Find the rejection region for the likelihood ratio test of $H_0 : \theta = \theta_0$ against $H_a : \theta \neq \theta_0$, where $\theta_0 > 0$.

13. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Show that the likelihood ratio test of $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$, where $\mu_0 > 0$, has a rejection region of the form

$$\{(x_1, \ldots, x_n) \in I\!R^n : \frac{\sqrt{n}|\bar{x} - \mu_0|}{\sqrt{\frac{1}{n-1}\sum_{j=1}^n (X_j - \bar{x})^2}} \geq t_{\frac{\alpha}{2}}(n-1)\},$$

where $c > 0$.

14. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2. Show that the likelihood ratio test of $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 \neq \sigma_0^2$, where $\sigma_0^2 > 0$, has a rejection region of the form

$$\{(x_1, \ldots, x_n) \in I\!R^n : \frac{\sum_{j=1}^n (x_j - \bar{x})^2}{n\sigma_0^2} \leq c_1 \text{ } or \text{ } \frac{\sum_{j=1}^n (x_j - \bar{x})^2}{n\sigma_0^2} \geq c_2\},$$

where $e^{-c_1} c_1 = e^{-c_2} c_2$ and $0 < c_1 < 1 < c_2$.

Tests

15. A company chip manufacturer claims that no more than 2 percent of the is sends out are defective. An electronics company, impressed by this claim, has purchased a large quantity of such ships. To determine if the manufacturer's claim can be taken literally, the company has decided to test a sample of 300 of these chips. If 10 of these 300 chips are found to be defective, should the manufacturer's claim rejected against the alternative hypothesis that the proportion of defective chips is higher than .02? What are the null and alternative hypothesis? Estimate the p–value. Would you reject the null hypothesis if the significance level is (a) $\alpha = 0.10$? (b) $\alpha = 0.05$? (c) $\alpha = 0.01$?

16. Do urban and rural households that display Christmas trees have the same preference for natural versus artificial trees? Let p_1 be the proportion of natural trees in urban households displaying a Christmas tree. Let p_2 be the proportion of natural trees in rural households displaying a Christmas tree. Test the hypothesis that rural household are more likely to choose natural Christmas trees. the survey responses show that 89 of the 261 urban households and 64 of the 160 rural households that displayed a tree chose a natural tree. What are the null and alternative hypothesis? Estimate the p–value. Would you reject the null hypothesis if the significance level is (a) $\alpha = 0.10$? (b) $\alpha = 0.05$? (c) $\alpha = 0.01$?

17. The breaking strengths of cables produced by a manufacturer have mean 1800 lb. It is claimed that a new manufacturing process will increase the mean breaking strength of the cables. To

test this hypothesis, a sample of 30 cables manufactured using the new process is tested, giving $\bar{Y} = 1850$ and $s = 100$. What are the null and alternative hypothesis? Estimate the p–value. Would you reject the null hypothesis if the significance level is (a) $\alpha = 0.10$? (b) $\alpha = 0.05$? (c) $\alpha = 0.01$?

18. To find out whether the inhabitants of two South Pacific islands may be regarded as having the same racial ancestry, an anthropologist determines the cephalic indices of six adult males from each island, getting $\bar{Y}_1 = 77.4$, $s_1 = 3.3$, $\bar{Y}_2 = 72.2$ and $s_2 = 2.1$. Assume the populations sampled are normal and have equal variances. Test whether the difference between the two sample means can reasonable be attributed to chance. What are the null and alternative hypothesis? Estimate the p–value. Would you reject the null hypothesis if the significance level is (a) $\alpha = 0.10$? (b) $\alpha = 0.05$? (c) $\alpha = 0.01$?

19. A gun–like apparatus has recently been designed to replace needles in administering vaccines. The apparatus can be set to inject different amounts of the serum, but because of random fluctuations the actual amount injected is normally distributed with a mean equal to the setting and with an unknown variance σ^2. It has been decided that the apparatus would be too dangerous to use if σ exceeds .10. If a random sample of 50 injections resulted in a sample standard deviation of .12, should use the new apparatus be discontinued? What are the null and alternative hypothesis? Estimate the p–value. Would you reject the null hypothesis if the significance level is (a) $\alpha = 0.10$? (b) $\alpha = 0.05$? (c) $\alpha = 0.01$?

20. An oil company claims that less than 20 % of all car owners have not tried its gasoline. Test this claim at the 0.01 level of significance if a random check reveals that 22 of 200 car owners have not tried the oil company's gasoline.

21. A market research firm supplies manufacturers with estimates of the retails sales of their products form samples of retail stores. Marketing managers are prone to look at the estimate and ignore sampling error. Suppose that a random sample of 75 stores this month shows mean sales of 52 units of a small appliance, with a standard deviation of 13 units. During the same month last year, a random sample of 53 stores gave mean sales of 49 units, with a standard deviation of 11 units. An increase from 49 to 52 is a rise of 6 %. The marketing is happy because sales are up 6 %. Give a 95 % confidence interval for the difference in mean number of units sold at all retail stores. Test at the 5 % level whether the mean number of units sold in a month is the same in this year and in last year. Estimate the p–value of the test. Should the manager be happy for the data?

22. Let Y equal the number of pounds of butterfat produced by a Holstein cow during the 305–day milking period following the birth of a calf. We assume that the distribution of Y is normal with mean μ and variance σ^2. The following data was obtained:

425	710	661	664	732	714	934	761	744
653	725	657	421	573	535	602	537	
405	874	791	721	849	567	468	975	

Test the null hypothesis $H_0 : \sigma^2 = 140^2$ against the alternative $H_a : \sigma^2 > 140^2$ at the level $\alpha = 0.05$. Do we accept or reject the null hypothesis at the level $\alpha = 0.05$? Find the p–value of the data.

23. A car company claims that its new experimental engine runs 29 minutes with one gallon of fuel. Test runs with an experimental engine it operated, respectively, for 24, 28, 21,23, 32, and 22 minutes with one gallon of fuel. Is there enough evidence at the .01 level to claim that the new

experimental engine runs less than 29 minutes? What are the null and alternative hypothesis? Determine the rejection region. Estimate the p value of the test.

24. Let Y equal the number of pounds of butterfat produced by a Holstein cow during the 305–day milking period following the birth of a calf. We assume that the distribution of Y is normal with mean μ and variance σ^2. The following data was obtained:

425	710	661	664	732	714	934	761	744
653	725	657	421	573	535	602	537	
405	874	791	721	849	567	468	975	

Test the null hypothesis $H_0 : \sigma^2 = 140^2$ against the alternative $H_a : \sigma^2 > 140^2$ at the level $\alpha = 0.05$. Determine the rejection region. Do we accept or reject the null hypothesis at the level $\alpha = 0.05$. Find the p–value of the data.

Other topics

25. A company claims to have 40% of the market for some product. You conduct a survey and find 38 out of 112 buyers (i.e. 34%) purchased this brand. Are these data consistent with the company's claim or is your survey result of 34% significantly different to the company's claim of 40%? Do a two-sided test at the level 0.05.

26. The temperature at which a thermostat goes off is normally distributed with variance σ^2. If the thermostat is to be tested five times, find $P(.85 \leq \frac{s^2}{\sigma^2} \leq 1.15)$.

27. An oil company claims that the sulfur content of its diesel fuel is at most .15 percent. To check this claim, the sulfur content of 40 randomly chosen samples were determined; the resulting sample mean and sample standard deviation were .162 and .040, respectively. What is the approximate probability that the sample mean would have been as high or higher than .162 if the company's claims were true? (Hint: use the t–distribution).

28. Students scores on exams given by a certain instructor have normal distribution with mean 70 and standard deviation 25. This instructor is about to give two exams, one for a class of size 15 and the other to a class of size 40. Approximate the probability that the average test score in the larger class exceeds that of the other class by over 4 points.

29. Twelve percent of the population is left handed. Find the probability that there are between 10 and 14 left–handers in a random sample of 100 members of this population. That is, find $P(10 \leq X \leq 14)$, where X is the number of left–handers in the sample.

30. Let $Y_1, \ldots, Y_n$ denote a random sample of size n from a uniform distribution on $(0, \theta)$, where $\theta > 0$. Find a constant a such that $a\bar{Y}$ is an unbiased estimator of θ. Find a constant b such that $bY_{(n)} = b \max(Y_1, \ldots, Y_n)$ is an unbiased estimator of θ. Find the mean square error of $a\bar{Y}$ and of $bY_{(n)}$ (for the a and b found before). Which estimator is prefered?

31. In a random sample of 300 persons eating lunch at a department store cafeteria, only 102 had dessert. If we use the sample proportion $\frac{102}{300} = 0.45$ as an estimate of the corresponding true proportion, with what confidence can we assert that out error is less than 0.05?

32. Let $Y_1, \ldots, Y_n$ denote a random sample from the density

$$f(y|\theta) = \begin{cases} \frac{1}{\theta} & \text{if } 0 < y < \theta, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$. Show that $\hat{\theta}_1 = \frac{n+1}{n} Y_{(n)}$ and $\hat{\theta}_2 = 2\bar{Y}$ are unbiased estimators of θ. Find the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$. Which estimator is preferred?

33. Given a random sample of size n from the density

$$f(y|\theta) = \begin{cases} \frac{my^{m-1}e^{-\frac{y^m}{\theta}}}{\theta} & \text{if } 0 < y, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is unknown parameter and $m > 0$ is known. Find a sufficient statistic of θ. Find the MVUE of θ.

34. Given a random sample of size n from the density

$$f(y|\theta) = \begin{cases} \frac{\alpha y^{\alpha-1}}{\theta^\alpha} & \text{if } 0 < y < \theta, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is unknown parameter and $\alpha > 0$ is known. Find a sufficient statistic of θ. Find the MVUE $\hat{\theta}_1$ of θ.

35. Let $Y_1, \ldots, Y_n$ denote a random sample from the density

$$f(y|\theta) = \begin{cases} \theta y^{\theta-1} & \text{if } 0 < y < 1, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$. Find the method of moments estimator of θ. Show that this estimator is consistent.

36. Given a random sample of size n from the density

$$f(y|\theta) = \begin{cases} \frac{my^{m-1}e^{-\frac{y^m}{\theta}}}{\theta} & \text{if } 0 < y, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is unknown parameter and $m > 0$ is known. Find the method of moments estimator of θ. Show that this estimator is consistent.

37. Let $Y_1, \ldots, Y_n$ be a random sample from

$$f(y|\theta) = \begin{cases} \theta y^{\theta-1}, & \text{if } 0 < y < 1, \\ 0 & \text{else.} \end{cases}$$

Find the maximum likelihood estimator of θ.

38. Let $Y_1, \ldots, Y_n$ be a random sample from the density $f(y|\theta) = \frac{(\ln\theta)^y}{\theta\ y!}$, $y = 0, 1, 2, \ldots$, $\theta > 1$. Find the maximum likelihood estimator of θ.

39. Given a random sample of size n from the density

$$f(y|\theta) = \begin{cases} e^{-(y-\theta)}, & \text{if } y \geq \theta, \\ 0 & \text{else.} \end{cases}$$

where $\theta \in I\!\!R$ is an unknown parameter. Find a sufficient statistic of θ. Find the maximum likelihood estimator of θ.

40. Given a random sample of size n from the density

$$f(y|\theta) = \begin{cases} \frac{5y^4}{\theta^5}, & \text{if } 0 \leq y \leq \theta, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is an unknown parameter. Find the maximum likelihood estimator of θ.

41. Let $Y_1, \ldots, Y_{10}$ be independent identically distributed random variables with a normal distribution with mean 3 and variance 40. Find b so that

$$P\{\bar{Y} \leq b\} = 0.95 \ .$$

42. Let $Y_1, \ldots, Y_{10}$ be independent identically distributed random variables with a normal distribution with mean zero and variance 4. Find b so that

$$\Pr\{\sum_{i=1}^{10} Y_i^2 \leq b\} = 0.05.$$

43. Suppose that a random sample of nine recently sold houses in a certain city resulted in a sample mean price of $122,000, with a sample standard deviation of $12,000. Give a 95 percent confidence interval for the mean price of all recently sold houses in this city.

44. If 132 of 200 male voters and 90 of 159 female voters favor certain candidate running for governor of Illinois. Find a 99 % confidence interval for the difference between the actual proportion of male and female who favor the candidate.

45. The capacities in ampere–hours of 10 batteries were recorded as follows:

140 136 150 144 148 152 138 141 143 151

(a) Estimate the population variance σ^2. (b) Compute a 99 percent confidence interval of σ^2.

46. Let $Y_1, \ldots, Y_n$ be a random sample from a normal distribution with mean zero and variance $\theta > 0$. (a) Find a sufficient statistic of θ. (b) Find the MVUE for θ.

47. Let $Y_1, \ldots, Y_n$ be a random sample from the density

$$f(y|\theta) = \begin{cases} \frac{2\theta^2}{y^3} & \text{if } y > \theta, \\ 0 & \text{else,} \end{cases}$$

where $\theta > 0$. (a) Find a sufficient statistic for θ. (b) Find the minimum variance unbiased estimator of θ. (c) Find the method of the moments estimator of θ.

48. Given a random sample $Y_1, \ldots, Y_n$ of size n from the density

$$f(y|\theta) = \begin{cases} \theta 4y^3 e^{-\theta y^4} & \text{if } y \geq 0, \\ 0 & \text{if } y < 0, \end{cases}$$

where $\theta > 0$ is unknown. Find the method of the moments estimator of θ.

49. Given a random sample of size n from the density

$$f(y|\theta) = \begin{cases} \theta y^{\theta-1} & \text{if } 0 < y, \\ 0 & \text{else.} \end{cases}$$

where $\theta > 0$ is unknown parameter. Find the maximum likelihood estimator of θ

50. Let $Y_1, \ldots, Y_n$ be a random sample from the density

$$f(y|\theta) = \begin{cases} \frac{3y^5 e^{-\frac{y^3}{\theta}}}{\theta^2} & \text{if } y > 0, \\ 0 & \text{else,} \end{cases}$$

where $\theta > 0$. (a) Find a sufficient statistic for θ. (b) Find the minimum variance unbiased estimator of θ.

51. Let $Y_1, \ldots, Y_n$ be a random sample from the density

$$f(y|\theta) = \begin{cases} \frac{4y^3}{\theta^4} & \text{if } 0 \le y \le \theta, \\ 0 & \text{else,} \end{cases}$$

where $\theta > 0$ is unknown. (a) Find the method of the moments estimator of θ. (b) Find the maximum likelihood estimator of θ.

52. Find the mean square error of $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$ as an estimator of σ^2, i.e. find

$$E[(s^2 - \sigma^2)^2].$$

Assume that $\{X_i\}$ are i.i.d.r.v.'s from a normal distribution, find the value of $t \in \mathbb{R}$, which minimizes the mean square error of $t \sum_{i=1}^n (x_i - \bar{x})^2$ as an estimator of σ^2, i.e. find the value of t which minimizes

$$E[(t \sum_{i=1}^n (x_i - \bar{x})^2 - \sigma^2)^2].$$

Hint:

$$\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2 - (\bar{x} - \mu)^2.$$

53. A r.v. X is said to have a lognormal distribution with parameters μ and σ^2 if $\ln X$ has a normal distribution with parameters μ and σ^2. Consider a random sample of size n from a lognormal distribution (μ, σ^2). Find the method of the moments estimators for μ and σ^2.

54. Consider a random sample of size n from an uniform distribution (θ_1, θ_2), $\theta_1 < \theta_2$. Find the method of the moments estimators for θ_1 and θ_2.

55. Let $X_1, \ldots, X_n$ be a random sample with pdf

$$f(x, \theta) = \frac{1}{\theta}, \quad 0 < x < \theta.$$

Find the method of moments estimator of θ and the MLE of θ. Find the mean and mean square error of both estimators. Which one should be preferred and why?

56. Let $X_1, \ldots, X_n$ be a random sample from a geometric distribution with parameter $0 < p < 1$, i.e. $P\{X = x\} = (1-p)^x p$, for $x = 0, 1, \ldots,$ Find the mle of p. Find the MVUE of p.

57. Given a random sample of size n from a population having the density

$$f(x, \theta) = \begin{cases} \theta x^{\theta - 1} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

where $\theta > 0$. Find the maximum likelihood estimator $\hat{\theta}$ of $\tau(\theta) = \theta$. Show that $\hat{\theta}^{-1}$ has a gamma distribution. Determine the parameters of this gamma distribution. Find $E_\theta[\hat{\theta}]$. Is $\hat{\theta}$ an unbiased estimator of θ? Find the MVUE of $\tau(\theta) = \theta$.

58. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on the interval $(-\theta, \theta)$, where $\theta > 0$. Show that the method of moments estimator of θ is not defined. Find the mle of θ.

59. Let $X_1, \ldots, X_n$ be a random sample from $f(x|\theta) = \frac{1}{\theta} m x^{m-1} e^{-x^m/\theta}$, $x > 0$, where $m > 0$ is known and $\theta > 0$ is unknown. Find the mle $\hat{\theta}$ of θ. Find the MVUE of $\tau(\theta) = \theta$.

60. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with parameter, $\theta > 0$. Find a sufficient statistic for θ. Which one of the next statistics are sufficient for θ: $T_1(X_1, \ldots, X_n) = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$, $T_2(X_1, \ldots, X_n) = \sum_{i=1}^n X_i$, $T_3(X_1, \ldots, X_n) = \sum_{i=1}^n X_i^2$? Why?

61. Let $X_1, \ldots, X_n$ be a random sample from a beta distribution with parameters α and β: $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$

$$f(x|\alpha,\beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find a sufficient statistic for (α, β).

62. Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ with μ and σ^2 unknown. Find the UMVUE of σ^p, where $p > 0$.

63. Let $X_1, \ldots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ with μ and σ^2 unknown. Find the UMVUE of σ^4. Find the variance of the UMVUE of σ^4.

64. Let $X_1, \ldots, X_n$ be a random sample from $f(x|\theta) = \frac{2x}{\theta^2}$, $0 < x < \theta$. Find a complete sufficient statistic for θ. Justify that the statistic is complete and sufficient. Find the UMVUE of θ. Find the UMVUE of θ^2.

65. Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution on $(0, \theta)$. Show that $X_{(n)}$ is a complete and sufficient statistic for θ. Find the UMVUE for θ^p, where $p > 0$.

66. Let $X_1, \ldots, X_n$ be a random sample from a exponential distribution with parameter $\theta > 0$. Find the CRLB for the unbiased estimators of θ^2. Find the UMVUE for θ^2.

67. Let $X_1, \ldots, X_n$ be a random sample from $f(x|\theta) = \frac{2\theta^2}{x^3}$, if $x > \theta > 0$. Find a complete sufficient statistic. Find a UMVUE for θ.

68. The price index values for 30 homes in thousand of dollars in a suburban area of the Northeast are:

$$0, 50, 56, 65, 72, 80, 82, 83, 99, 101, 110, 112, 117, 120, 140$$

$$144, 145, 150, 180, 201, 210, 220, 240, 290, 309, 320, 325, 400, 500, 507$$

Find the mean price of a house. Find a 95 % level confidence interval for the average price of a house.

69. In a random sample of 300 persons eating lunch at a department store cafeteria, only 102 had dessert. Use the previous data to find a 95 % level confidence interval for the average number of people which take dessert at lunch.

70. Let $X_1, \ldots, X_n$ be a random sample froma $N(\theta, 1)$. A 95% confidence interval for θ is $\bar{x} \pm 1.96/\sqrt{n}$. Let p be the additional probability that an additional independent observation, X_{n+1} will fall in this interval. Is p greater, less than, equal to .95? Justify your answer.